

1 (a) $\lim_{n \rightarrow \infty} a_n = L$ means $\forall \epsilon > 0 \exists N \in \mathbb{N}$ so

n large enough $\rightarrow \forall n > N_\epsilon, |a_n - L| < \epsilon$. a_n close to L

(b) Abel's Test If $a_n > 0 \forall n$ and $a_{n+1} \leq a_n \forall n \geq 1$

and $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} n a_n = 0$. Thus,

if $\lim_{n \rightarrow \infty} n a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges (to ∞).

(c) If there exists a function $f: [1, \infty) \rightarrow \mathbb{R}$ such that

$x < y \Rightarrow f(x) \geq f(y)$ and $\int_1^{\infty} f(x) dx \leq B$ then

if $a_n = f(n) \forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} a_n$ converges.

(d) $\lim_{x \rightarrow a^+} f(x) = \infty$ means $\forall M > 0 \exists \delta_M > 0$ so

$a < x < a + \delta_M \Rightarrow f(x) > M$

x close to a & on the right

$f(x)$ large

2 Let $S \subset \mathbb{R}$ be non-empty and bounded above. Then S has a least upper bound i.e. if $\exists B \in \mathbb{R}$ so $\forall x \in S, x \leq B$, then $\exists \beta \in \mathbb{R}$ so $\forall x \in S, x \leq \beta$ & $\forall \epsilon > 0 \exists y \in S$

so $\beta - \epsilon < y \leq \beta$.

If $0 < c < 1$ let $a_n = c^n, n \in \mathbb{N}$. Then $a_{n+1} = c^{n+1} = c \cdot c^n = c \cdot a_n$

$\Rightarrow \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} c \cdot a_n = c \lim_{n \rightarrow \infty} a_n$ (*) Since (a_n) is decreasing

$(a_{n+1} < a_n)$ and bounded below $(0 < a_n) \Rightarrow \lim_{n \rightarrow \infty} a_n$ exists. Let it be L . Then $L = \lim_{n \rightarrow \infty} a_{n+1} = cL \Rightarrow L = cL \Rightarrow L(1-c) = 0 \Rightarrow L = 0$

3) Let $\epsilon > 0$ be given.

Since $\lim_{x \rightarrow a^+} f(x) = L$, $\exists \delta_1 > 0$ so $a < x < a + \delta_1 \Rightarrow |f(x) - L| < \frac{\epsilon}{2}$

Since $\lim_{x \rightarrow a^+} g(x) = M$, $\exists \delta_2 > 0$ so $a < x < a + \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2} \therefore$

Let $\delta_\epsilon = \min\{\delta_1, \delta_2\}$. If $a < x < a + \delta_\epsilon$ then $a < x < a + \delta_1$ and $a < x < a + \delta_2$

$$\begin{aligned} \text{so } |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore $\lim_{x \rightarrow a^+} f(x) + g(x) = L + M$ //

4) (a) Let $\epsilon > 0$ be given. Working back we want

$$\begin{aligned} \left| \frac{4n+1}{2n+1} - 2 \right| < \epsilon &\iff \left| \frac{4n+1 - (4n+2)}{2n+1} \right| < \epsilon \\ &\iff \frac{|-1|}{|2n+1|} < \epsilon \iff \frac{1}{2n+1} < \epsilon \\ &\iff 2n+1 > \frac{1}{\epsilon} \quad (\text{since } \epsilon > 0 \text{ and } n \geq 1) \\ &\iff n > \frac{1}{2\epsilon} - \frac{1}{2} \end{aligned}$$

so let $N_\epsilon > \frac{1}{2\epsilon} - \frac{1}{2}$. Then if $n \geq N_\epsilon \Rightarrow n > \frac{1}{2\epsilon} - \frac{1}{2}$

then $\left| \frac{4n+1}{2n+1} - 2 \right| < \epsilon$. Hence $\lim_{n \rightarrow \infty} \frac{4n+1}{2n+1} = 2$.

OR $\lim_{n \rightarrow \infty} \frac{4n+1}{2n+1} = \lim_{n \rightarrow \infty} \frac{4 + \frac{1}{n}}{2 + \frac{1}{n}} = \frac{4+0}{2+0} = 2$ //

(b) $a_n = \frac{2^n}{n!} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)n!} \times \frac{n!}{2^n} = \frac{2}{n+1}$

d $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

5) $f(x) = 2x + \frac{1}{x}$, $f(1) = 2 + \frac{1}{1} = 3$

Let $\epsilon > 0$ be given. We want $|f(x) - f(1)| < \epsilon$

$\Leftrightarrow \left| 2x + \frac{1}{x} - 3 \right| < \epsilon$

$\Leftrightarrow \left| \frac{2x^2 + 1 - 3x}{x} \right| < \epsilon \Leftrightarrow \left| \frac{(2x-1)(x-1)}{x} \right| < \epsilon$

$\Leftrightarrow \frac{|2x-1||x-1|}{|x|} < \epsilon$

Let $\delta_1 = \frac{1}{2}$. Then $|x-1| < \frac{1}{2} \Rightarrow |x-1| < |x-1| = |x-1| < \frac{1}{2} \Rightarrow \frac{1}{2} < |x| \Rightarrow \frac{2}{|x|} > 1$

then $|2x-1| = |2(x-1)+1| \leq 2|x-1|+1 \leq 2 \cdot \frac{1}{2} + 1 = 2$

Hence if $2 \cdot 2 |x-1| < \epsilon$ we would have $|f(x) - f(1)| < \epsilon$.

so let $\delta_\epsilon = \min \left\{ \frac{1}{2}, \frac{\epsilon}{4} \right\} \dots \Rightarrow \lim_{x \rightarrow 1} f(x) = f(1)$.

Therefore $f(x)$ is continuous at $x=1$.

Since $f(x) = 2x + \frac{1}{x}$, if x is near 0 and positive, $f(x)$ is large and positive, i.e. $\lim_{x \rightarrow 0^+} f(x) = +\infty$. Similarly

if $x \rightarrow 0^-$, $f(x) \rightarrow -\infty$. If $x \rightarrow \pm\infty$, $\frac{1}{x} \rightarrow 0$, Hence

$f(x) - 2x = \frac{1}{x} \rightarrow 0$ and $f(x)$ goes to $\pm\infty$ like $2x$.

