

$$\textcircled{1} \quad \sum_{n=2}^5 \left( \frac{3}{n(n-1)} + \frac{2}{4^n} \right) = 3 \left( \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} \right) + 2 \left( \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \frac{1}{4^5} \right)$$

$$= \frac{6569}{2560} = 2.566..$$

$$S = \sum_{n=2}^{\infty} \left( \frac{3}{n(n-1)} + \frac{2}{4^n} \right) = 3 \sum_{n=2}^{\infty} \frac{1}{n(n-1)} + 2 \left( \sum_{n=0}^{\infty} \frac{1}{4^n} - \left(1 + \frac{1}{4}\right) \right)$$

But  $\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$  so  $\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \dots + \left( \frac{1}{N-1} - \frac{1}{N} \right)$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \lim_{N \rightarrow \infty} \left( 1 - \frac{1}{N} \right) = 1 = S_1 \quad (\text{telescoping sum})$$

Hence since  $S_2 = \frac{1}{1 - 1/4}$  (geometric series)  
 $= \frac{4}{3}$

$$\textcircled{*} \Rightarrow S = 3 \cdot 1 + 2 \left( \frac{4}{3} - \frac{5}{4} \right) = \frac{19}{6} //$$

Claim  $(\log n) < (n^d) < (n) < (2^n) < (e^n) < (n!) < (n^n)$   
 (a) (b) (c) (d) (e) (f)

(a)  $\lim_{n \rightarrow \infty} \frac{\log n}{n^d} = \lim_{m \rightarrow \infty} \frac{\log(m^{1/d})}{m}$   
 $m = n^d = \lim_{m \rightarrow \infty} \left( \frac{1}{d} \right) \frac{\log m}{m} = \frac{1}{d} \cdot 0 = 0.$

(b)  $\lim_{n \rightarrow \infty} \frac{n^d}{n} = \lim_{n \rightarrow \infty} \frac{1}{n^{1-d}} = 0$  since  $1-d > 0$

[Proof: given  $\epsilon > 0$  choose  $N_{\epsilon}$  so  $\frac{1}{N_{\epsilon}^{1-d}} < \epsilon \Leftrightarrow N_{\epsilon} > \epsilon^{\frac{1}{1-d}}$   
 then  $n \geq N_{\epsilon} \Rightarrow \frac{1}{n} \leq \frac{1}{N_{\epsilon}} \Rightarrow \frac{1}{n^{1-d}} \leq \frac{1}{N_{\epsilon}^{1-d}} < \epsilon.$ ]

(c)  $\lim_{n \rightarrow \infty} \frac{n}{2^n} = \lim_{n \rightarrow \infty} a_n$ ;  $\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \left(1 + \frac{1}{n}\right) \cdot \frac{1}{2} \leq 1 \Rightarrow (a_n)$  is decreasing + bounded below by 0 so  $\lim_{n \rightarrow \infty} a_n = L$  exists. But

$$a_{n+1} = \left(1 + \frac{1}{n}\right) \frac{1}{2} a_n \Rightarrow L = (1+0) \frac{1}{2} L \Rightarrow L = 0. \text{ Hence}$$

(d)  $\lim_{n \rightarrow \infty} \frac{n}{e^n} = 0.$   
 $e > 2 \Rightarrow \left(\frac{2}{e}\right) < 1 \Rightarrow \lim_{n \rightarrow \infty} \left(\frac{2}{e}\right)^n = 0$

(e)  $\lim_{n \rightarrow \infty} \frac{n^n}{n!} = 0$  as  $n^n$  term of the conv. exponential series.

(f)  $\lim_{n \rightarrow \infty} \frac{n!}{n^n}$ ;  $a_n = \frac{n!}{n^n} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$   
 $= \frac{(n+1) n^n}{(n+1)^n (n+1) \cdot 1}$   
 $= \frac{1}{(1 + \frac{1}{n})^n} \rightarrow \frac{1}{e} < 1$

Hence  $(a_n)$  is decreasing and pos. Thus

$\lim_{n \rightarrow \infty} a_n = L$  say so  $a_{n+1} = \frac{1}{(1 + \frac{1}{n})^n} a_n \Rightarrow L = \frac{1}{e} L$

$\Rightarrow L = 0$ .

Question can you find a sequence going to infinity slower than  $(a_n)$   
 all  $a_n = \underbrace{\log \log \log \dots (n)}_{\text{any finite number of logs}}$ ?  
 or faster than all  $(b_n)$ :  $b_n = \left( (n^n)^n \right)^{\dots n}$

(3) (a)  $a_n = \frac{2}{n+1}$   $b_n = \frac{1}{n}$ : Since  $a_n \geq b_n$   
 $\Leftrightarrow \frac{2}{n+1} \geq \frac{1}{n}$   
 $\Leftrightarrow 2n \geq n+1 \Leftrightarrow n \geq 1$

$\sum_1^\infty (b_n)$   $\mathcal{D}$  so does  $\sum_1^\infty (a_n)$  by Comparison  
 OR: Since  $(a_n)$  is pos. and decreasing +  $n a_n = \frac{2n}{n+1} \rightarrow 2 \neq 0 \Rightarrow \mathcal{D}$

by Abel's test.

(b)  $a_n = \frac{(\log n)^n}{n^n} \Rightarrow a_n = \frac{\log n}{n} \rightarrow 0 < 1$  so  $\sum_1^\infty a_n \mathcal{C}$  by Cauchy.

(c)  $a_n = \frac{n!}{e^n}$   $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} = \frac{n+1}{e} \geq \frac{3}{e} > 1 \forall n \geq 2$

so by D'Alembert's (Ratio) test I  $\sum_1^\infty a_n \mathcal{D}$ .

(d)  $a_n = \frac{3^n (n!)^2}{(2n)!} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{3^{n+1} (n+1)! (n+1)! (2n)!}{(2(n+1))! 3^n n! n!}$   
 $= \frac{3^{\frac{3}{2}} (n+1)(n+1)}{(2n+2)(2n+1)} = \frac{3}{2} \frac{(n+1)}{(2n+1)} = \frac{3}{2} \left( \frac{1 + \frac{1}{n}}{2 + \frac{1}{n}} \right)$   
 $\rightarrow \frac{3}{4} < 1$ .  $\sum_{n=1}^\infty a_n \mathcal{C}$ .

Hence by D'Alembert's (Limit Ratio) test II  $\sum_{n=1}^\infty a_n \mathcal{C}$ .

(4) Just notice that  $a_n \geq 0 \Rightarrow S_{n+1} = S_n + a_{n+1} \geq S_n$  so  $(S_n)$  is increasing. Hence  $S_n \rightarrow S \in \mathbb{R} \Leftrightarrow (S_n)$  is bounded above.  
 (clear  $S_n = a_1 + a_2 + \dots + a_n \forall n \in \mathbb{N}$ )