Math252-03A Assignment 1 Solutions

(1) $S = \{ 2 - \frac{1}{n^2} : n = 1,2,3.. \}$

Since $0 < \frac{1}{n^2} \le 1 \quad \forall n \in \mathbb{N}$, $1 \le 2 - \frac{1}{n^2} < 2$

$$\left.\begin{aligned} -0 &> -\tfrac{1}{n^2} \ge -1 \\ 2 &> 2 - \tfrac{1}{n^2} \ge 2 - 1 = 1 \end{aligned}\right\} \Uparrow$$

So 1 is a lower bound and 2 an upper bound for S.

Since $1 = 2 - \frac{1}{1^2}$ is in S, if $1 < m$ then m cannot be a lower bound for S, hence $1 = glb(S)$.

If $M = 2 - \varepsilon$ is an upper bound for S for some $\varepsilon > 0$, then if $N_\varepsilon > \frac{1}{\sqrt{\varepsilon}} \Rightarrow \frac{1}{N_\varepsilon^2} < \varepsilon$ so

$$M = 2 - \varepsilon < 2 - \frac{1}{N_\varepsilon^2} \in S,$$ so M is not an u.bound for S. Hence $2 = lub(S)$. //

(2) $$\lim_{n\to\infty} \left(\frac{2n^2 - n + 1}{n^2 + 6} + \frac{4n+1}{n} \right)$$

$$= \lim_{n\to\infty} \frac{2n^2 - n + 1}{n^2 + 6} + \lim_{n\to\infty} \frac{4n+1}{n}$$ (lt. of a sum = sum of lts.)

$$= \lim_{n\to\infty} \frac{2 - \frac{1}{n} + \frac{1}{n^2}}{1 + \frac{6}{n^2}} + \lim_{n\to\infty} \left(\frac{4n}{n} + \frac{1}{n} \right)$$ ($\div$ by n^2 or n)

$$= \frac{\lim_{n\to\infty} 2 - \frac{1}{n} + \frac{1}{n^2}}{\lim_{n\to\infty} 1 + \frac{6}{n^2}} + \lim_{n\to\infty} 4 + \lim_{n\to\infty} \frac{1}{n}$$ (lt. of a quotient)

$$= \frac{2 - 0 + 0}{1 + 0} + 4 + 0$$

$$= 6. //$$

(3) $a_n := \dfrac{\sin(n) + n^2}{n^2} = \dfrac{\sin(n)}{n^2} + 1$

$-1 \leq \sin(n) \leq +1 \qquad \forall n \in \mathbb{N} \Rightarrow$

$-\frac{1}{n^2} \leq \frac{\sin(n)}{n^2} \leq \frac{1}{n^2}$

$\Rightarrow 1 - \frac{1}{n^2} \leq \frac{\sin(n)}{n^2} + 1 \leq 1 + \frac{1}{n^2}$

and $\lim_{n\to\infty}(1 - \frac{1}{n^2}) = 1 - 0 = 1 = \lim_{n\to\infty}(1 + \frac{1}{n^2})$.

$\therefore$ by the Sandwich Theorem, $\lim_{n\to\infty} a_n = 1$ also.

(4) Let $\varepsilon > 0$ be given. Want $N_\varepsilon \in \mathbb{N}$ so that

$|a_n - 2| < \varepsilon$. This is true $\Leftrightarrow$

$\left|\frac{2n+1}{n-1} - 2\right| < \varepsilon \Leftrightarrow \left|\frac{2n+1-2n+2}{n-1}\right| < \varepsilon$

$\Leftrightarrow \frac{3}{n-1} < \varepsilon \Leftrightarrow \frac{3}{\varepsilon} + 1 < n$

so chose N_ε as any whole number $> \frac{3}{\varepsilon} + 1$

Then if $n \geq N_\varepsilon$ we have $\frac{3}{\varepsilon} + 1 < N_\varepsilon \leq n \Rightarrow$

$\frac{3}{\varepsilon} + 1 < n \Rightarrow |a_n - 2| < \varepsilon$, by the chain of $\Leftrightarrow$'s.

Hence $\lim_{n\to\infty} a_n = 2$.

(5) Given $0 < c < 1$ and $a_n = c^n \Rightarrow a_{n+1} = c^{n+1}$
$= c.c^n$
$= c.a_n < 1.a_n$

so $0 < a_{n+1} < a_n \quad \forall n \in \mathbb{N}$.

Hence the sequence (a_n) is decreasing and bounded below.

Therefore, by the monotonely decreasing seq Theorem,

$L = \lim_{n\to\infty} a_n$ exists.

But $a_{n+1} = c.a_n \Rightarrow L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} c.a_n$
$= c \lim_{n\to\infty} a_n$
$= c.L$

Hence $L = c.L \Rightarrow 0 = (1-c).L$

But $1 - c \neq 0$. Thus $L = 0$ & so $\lim_{n\to\infty} c^n = 0$