① $\frac{1}{1-x} > \frac{1}{2} \Rightarrow 1-x > 0 \Rightarrow 1 > x \Rightarrow x < 1$

Then $2 > 1-x \Rightarrow 1 > -x \Rightarrow x > -1 \Rightarrow -1 < x$

$\Big\}\ x \in (-1, 1)$.

② Let $\varepsilon > 0$ and $|x| \le \varepsilon$.

If $x \geq 0$, $-\varepsilon < 0 \le x = |x| \le \varepsilon \Rightarrow -\varepsilon \le x \le \varepsilon$.

If $x < 0$, $-x = |x| \le \varepsilon \Rightarrow -\varepsilon \le x < 0 \le \varepsilon \Rightarrow -\varepsilon \le x \le \varepsilon$.

$\therefore$ in all cases $-\varepsilon \le x \le \varepsilon$.

③ $$\lim_{n\to\infty}\left(\frac{n^2+1}{n^2+2}\right)\left(2-\frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n^2}}{1+\frac{2}{n^2}}\right)\lim_{n\to\infty}\left(2-\frac{1}{n}\right) \quad (*)$$

(* since lim. prod = prod. lims.)

$$= \left(\frac{\lim_{n\to\infty}(1+\frac{1}{n^2})}{\lim_{n\to\infty}(1+\frac{2}{n^2})}\right)\left(\lim_{n\to\infty} 2 - \lim_{n\to\infty}\frac{1}{n}\right) \quad (**)$$

(** lim. quot. = quot. lims. + lim. diff = diff. lims)

$$= \left(\frac{1+0}{1+0}\right)(2-0) \quad (***)$$

$$= 2$$

(*** where lim sum = sum. lims has been used to derive e.g. $\lim_{n\to\infty}(1+\frac{1}{n^2}) = \lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{n^2} = 1+0$ and

where $\lim_{n\to\infty}(\text{constant}) = \text{constant}$

$\lim_{n\to\infty}\frac{1}{n^2} = (\lim_{n\to\infty}\frac{1}{n})^2 = 0^2 = 0$.)

④ $\forall n \in \mathbb{N}$ $-1 \le \sin(n) \le 1 \Rightarrow \frac{-n}{n^2+1} \le \frac{n\sin(n)}{n^2+1} \le \frac{n}{n^2+1}$

But if $a_n = \frac{n}{n^2+1} = \frac{1/n}{1+1/n^2}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{1}{n^2}} = \frac{0}{1+0} = 0$ and $\lim_{n\to\infty}(-a_n) = -\lim_{n\to\infty} a_n = -0 = 0$.

and $-a_n \le \frac{n\sin(n)}{n^2+1} \le a_n$, so by the sandwich theorem,

$$\lim_{n\to\infty}\frac{n\sin(n)}{n^2+1} = 0.$$

(5) Given $\varepsilon > 0$. We want N_ε so $|a_n - 3| < \varepsilon$. This is true $\iff$

$$\left| \frac{6n-2}{2n+1} - 3 \right| < \varepsilon$$

$$\iff \left| \frac{6n-2-6n-3}{2n+1} \right| < \varepsilon$$

$$\iff \left| \frac{-5}{2n+1} \right| < \varepsilon \iff \frac{5}{2n+1} < \varepsilon$$

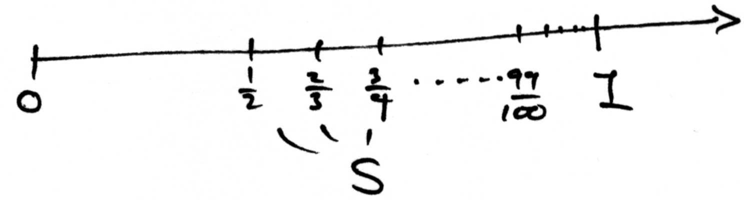
$$\iff \frac{5}{\varepsilon} < 2n+1$$

$$\iff \frac{5}{2\varepsilon} - \frac{1}{2} < n \qquad \boxtimes$$

So choose N_ε as any $\mathbb{N}$ with $\frac{5}{2\varepsilon} - \frac{1}{2} < N_\varepsilon$

Then if $n \geq N_\varepsilon$ we have $\frac{5}{2\varepsilon} - \frac{1}{2} < N_\varepsilon \leq n \Rightarrow \boxtimes$ holds $\therefore$ $|a_n - 3| < \varepsilon$. Thus $\lim_{n\to\infty} a_n = 3$ is proved.

(6) $S = \left\{ \frac{n}{n+1} \mid n \in \mathbb{N} \right\}$

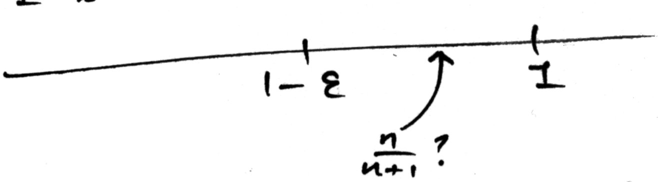


Firstly $\forall n \in \mathbb{N}$ $\frac{1}{2} \leq \frac{n}{n+1}$

To see this, the given inequality is true $\iff n+1 \leq 2n \iff 1 \leq n$ which is true. Also $\frac{1}{2} \in S$. Therefore $\frac{1}{2}$ is a l.b. for S for which no strictly greater l.b can $\exists$ (since $\frac{1}{2}$ would be smaller). Hence $\frac{1}{2} = \text{glb}(S)$.

Now $\forall n$, $\frac{n}{n+1} < 1$. This is true since $\frac{n}{n+1} < 1 \iff n < n+1 \iff 0 < 1$.

So 1 is an u.b. If $\varepsilon > 0$ is given



$$1 - \varepsilon < \frac{n}{n+1} < 1 \iff 1 - \varepsilon < \frac{n+1-1}{n+1} < 1$$

$$\iff 1 - \varepsilon < 1 - \frac{1}{n+1} < 1$$

$$\iff -\varepsilon < -\frac{1}{n+1} < 0$$

$$\iff \varepsilon > \frac{1}{n+1} > 0 \iff 0 < \frac{1}{n+1} < \varepsilon$$

& we can always find such an n. Hence 1 must be the lub(S), since any strictly smaller ~~u.b~~ number cannot be an u.b. //