(1) $a_n = \frac{1}{2n(n+1)} + \frac{5}{3^n}$

To 4 terms sum $= a_2 + a_3 + a_4 + a_5$

$= \ldots = \frac{481}{486}$

To infinity sum $= \sum_{n=2}^{\infty}\left(\frac{1}{2n(n+1)} + \frac{5}{3^n}\right)$

$= \frac{1}{2}\left(\left\{\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\right\} - \frac{1}{1.2}\right) + 5\left(\left\{\sum_{n=0}^{\infty}\frac{1}{3^n}\right\} - \frac{1}{3^0} - \frac{1}{3^1}\right)$

$= \frac{1}{2}\left(S_1 - \frac{1}{2}\right) + 5\left(S_2 - \frac{4}{3}\right)$ $\boxtimes$

where $S_1 = \sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ and $S_2 = \sum_{n=0}^{\infty}\frac{1}{3^n}$

now if $w_n = \sum_{j=1}^{n}\frac{1}{j(j+1)} = \sum_{j=1}^{n}\left(\frac{1}{j} - \frac{1}{j+1}\right)$

$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1}$

So $S_1 = \lim_{n\to\infty} w_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1.$

Also $S_2 = \frac{1}{1 - \frac{1}{3}}$ since it is a geometric series with ratio $r = \frac{1}{3}$.

$= \frac{3}{2}$

Hence, by $\boxtimes$ above the sum to infinity is

$S = \frac{1}{2}\left(1 - \frac{1}{2}\right) + 5\left(\frac{3}{2} - \frac{4}{3}\right)$

$= \frac{13}{12}$ //

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(2) (i) since $\lim_{n\to\infty}\frac{n}{n^2} = \lim_{n\to\infty}\frac{1}{n} = 0 \Rightarrow (n) < (n^2)$.

(ii) since $\lim_{n\to\infty}\frac{2^n}{e^n} = \lim_{n\to\infty}\left(\frac{2}{e}\right)^n = 0 \Rightarrow (2^n) < (e^n)$.

(iii) for $0 < \alpha < 1$ $\lim_{n\to\infty}\left(\frac{n^\alpha}{n}\right) = \lim_{n\to\infty}\frac{1}{n^{1-\alpha}} = 0 \Rightarrow (n^\alpha) < (n)$.

(iv) Consider the series $\sum_{n=1}^{\infty} \frac{e^n}{n!} = \sum_{n=1}^{\infty} a_n$

Then $\frac{a_{n+1}}{a_n} = \frac{e^{n+1}}{(n+1)!} \times \frac{n!}{e^n} = \frac{e}{n+1} \to 0 < 1$. Hence by the limit ratio test, the series C $\Rightarrow a_n \to 0$. Hence $(e^n) < (n!)$.

(v) Let $a_n = \frac{n!}{n^n} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$

$$= \frac{(n+1) \cdot n^n}{(n+1)(n+1)^n}$$

$$\overset{\otimes}{=} \frac{1}{(1+\frac{1}{n})^n} < 1$$

Hence (a_n) is decreasing and bdd below by 0.

Hence $\lim_{n\to\infty} a_n = L$ exists. By $\otimes$ $a_{n+1} \times (1+\frac{1}{n})^n = a_n$

$\Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (1+\frac{1}{n})^n \times \lim_{n\to\infty} a_n$

$L = e \times L \Rightarrow L = 0$. Hence $(n!) < (n^n)$.

(vi) Let $f(x) = \frac{\log x}{x}$ for $x > 0$. Then

$$f'(x) = \frac{x/x - \log x}{x^2} = \frac{1 - \log x}{x^2}$$ by the quotient rule.

But $\lim_{x\to\infty} \log x = \infty$ & $\log x > 1$ $\forall x > e$. Hence $f'(x) < 0$ $\forall x > e$ so if $a_n = \frac{\log n}{n}$, a_n is decreasing & bdd below by 0. Hence $\lim_{n\to\infty} a_n = L$ exists in $\mathbb{R}$.

But then the subsequence $\lim_{n\to\infty} a_{2n} = L$ also. But

$$a_{2n} = \frac{\log(2n)}{2n} = \frac{1}{2}\frac{\log 2 + \log n}{n} = \frac{\log 2}{2n} + \frac{1}{2}\frac{\log n}{n}$$

$\Rightarrow L = 0 + L/2 \Rightarrow L = 0$. Hence $(\log n) < (n)$

(vii) The proofs of the remaining orderings are similar! Ans:

$(\log\log n) < (\log n) < (n^\alpha) < (n) < (n^2) < (2^n) < (e^n) < (n^n)$

How about $(n \log n)$ and $(e^{\frac{n}{\log n}})$?

(3) (i) $a_n = \dfrac{n^3}{n^3-4}$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{1}{1-\frac{4}{n^3}}$

$= \dfrac{1}{1-0} = 1 \neq 0 \Rightarrow \mathcal{D}.$

(ii) $a_n = \dfrac{\sqrt{n}}{n\lg n + 1}$ $\quad b_n = \dfrac{1}{n}$

$a_n \geq b_n \iff \dfrac{\sqrt{n}}{n\lg n+1} \geq \dfrac{1}{n}$

$\iff n^{3/2} \geq n\lg n + 1$

$\iff 1 \geq \dfrac{\lg n}{n^{1/2}} + \dfrac{1}{n^{3/2}} \to 0$

so $\exists N_1$ so $a_n \geq b_n$ $\forall n \geq N_1$ (actually $N_1 = 1$ is o.k.!)

But $\sum_1^\infty b_n$ $\mathcal{D}$. $\therefore$ by the comparison test, $\sum_{n=1}^\infty a_n$ $\mathcal{D}$ also.

(iii) $a_n = \dfrac{2^n}{3^n+1}$, $b_n = \dfrac{2^n}{3^n} = \left(\dfrac{2}{3}\right)^n$

Then $a_n < b_n$ $\forall n \in \mathbb{N}$ & $\sum_{n=1}^\infty b_n$ is a geometric series with ratio $r = \frac{2}{3} < 1$ $\therefore$ $\mathcal{C} \Rightarrow \sum_{n=1}^\infty a_n$ $\mathcal{C}$ also by the comparison test.

(4) $\dfrac{a_{n+1}}{a_n} = \dfrac{c^{n+1}}{c^n} = c$ $\forall n \geq 1$. Hence $\overset{\boxtimes}{a_{n+1} = c.a_n} < a_n$ since $0 < c < 1$.

$\therefore$ (a_n) is decreasing & $0 < a_n$ $\forall n \Rightarrow$ bdd below.

Using $\boxtimes$, $\lim_{n\to\infty} a_{n+1} = c \lim_{n\to\infty} a_n$ & if $L = \lim_{n\to\infty} a_n$,

$L = c.L \Rightarrow (1-c)L = 0 \Rightarrow L = 0.$

Again $\dfrac{a_{n+1}}{a_n} = c < 1 \Rightarrow$, by the ratio test $\sum_{n=1}^\infty a_n$ $\mathcal{C}$.

Now if $w_n = 1 + c + c^2 + \dots + c^{n-1}$ $\Rightarrow$

$c.w_n = \quad c + c^2 + \dots\ c^{n-1} + c^n$

$(1-c)w_n = 1 - c^n \Rightarrow w_n = \dfrac{1-c^n}{1-c}.$

$\therefore \sum_{n=1}^\infty c^n = \lim_{n\to\infty} w_n = \lim_{n\to\infty} \dfrac{1-c^n}{1-c} = \dfrac{1-0}{1-c} = \dfrac{1}{1-c}$ //