Part B (2008 Test 2) math101 Solutions

1 (a)
$$\int_1^2 \left(x^3 + 2x^2 + \frac{1}{x}\right)dx = \left(\frac{x^4}{4} + \frac{2x^3}{3} + \ln|x|\right)\Big|_1^2$$

$$= \left(\frac{2^4}{4} + \frac{2.2^3}{3} + \ln(2)\right) - \left(\frac{1^4}{4} + \frac{2.1^3}{3} + \ln(1)\right)$$

$$= \frac{101}{12} + \ln(2)$$

(b) Let $u = 1 - x^2 \Rightarrow du = -2x\,dx$

So
$$\int \frac{2x\,dx}{\sqrt{1-x^2}} = \int \frac{-1}{\sqrt{u}}\,du = -\int u^{-\frac{1}{2}}\,du$$

$$= -2u^{\frac{1}{2}} + C$$

$$= -2\sqrt{1-x^2} + C$$

(c) $I = \int \frac{x}{(x-1)(x-2)}\,dx$. Let $\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

$$\Rightarrow \quad x = A(x-2) + B(x-1)$$

$x = 1 \Rightarrow \quad 1 = A(-1) + 0 \Rightarrow A = -1$

$x = 2 \Rightarrow \quad 2 = 0 + B(1) \Rightarrow B = 2$

$$\therefore \quad I = \int \left(\frac{-1}{x-1} + \frac{2}{x-2}\right)dx$$

$$= -\int \frac{dx}{x-1} + 2\int \frac{dx}{x-2} = -\ln(|x-1|) + 2\ln(|x-2|) + C$$

$$= \ln\left(\frac{|x-2|^2}{|x-1|}\right) + C$$

1 (d) $\cos(2x) = 2\cos^2 x - 1 \Rightarrow$

$$\cos^2(x) = \frac{1}{2} + \frac{\cos(2x)}{2} \quad \boxtimes$$

so $$\cos^4(x) = \frac{1}{4}\left(1+\cos(2x)\right)^2$$

$$= \frac{1}{4}\left(1 + 2\cos(2x) + \cos^2(2x)\right)$$

using $\boxtimes$ with $x \to 2x$: $$= \frac{1}{4}\left(1 + 2\cos(2x) + \frac{1}{2} + \frac{\cos(4x)}{2}\right)$$

$$= \frac{1}{8}\left(2 + 4\cos(2x) + 1 + \cos(4x)\right)$$

$$\therefore \int_0^{\pi} \cos^4 x\,dx = \frac{1}{8}\int_0^{\pi} (3 + 4\cos(2x) + \cos(4x))\,dx$$

$$= \frac{1}{8}\left[3x \quad \frac{4\sin(2x)}{2} + \frac{\sin(4x)}{4}\right]_0^{\pi}$$

$$= \frac{1}{8}\left[3\pi - 3.0\right] \quad \text{since } \sin(0) = \sin(2\pi) = \sin(4\pi) = 0$$

$$= \frac{3\pi}{8}$$

2 (a) $$|f(x)| = |x|\,|x^2+1|^{2/3}\,|x+4|^{-1} \Rightarrow \quad (\log = \ln)$$

$$\log|f(x)| = \log|x| + \frac{2}{3}\log|x^2+1| - \log|x+4|$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = \frac{(x)'}{x} + \frac{2}{3}\frac{(x^2+1)'}{x^2+1} - \frac{(x+4)'}{x+4}$$

$$= \frac{1}{x} + \frac{4x}{3(x^2+1)} - \frac{1}{x+4}$$

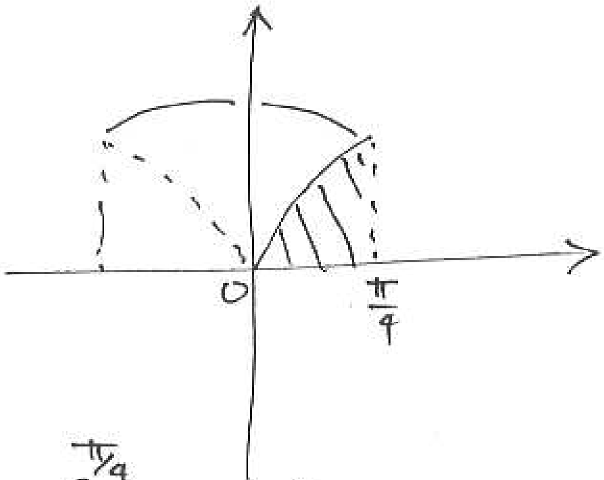
$$\Rightarrow \quad f'(x) = f(x)\left[\frac{1}{x} + \frac{4x}{3(x^2+1)} - \frac{1}{x+4}\right]$$

This is well defined if $x \neq 0$ and $x \neq -4$.

$$f'(1) = f(1)\left[\frac{1}{1} + \frac{4}{6} - \frac{1}{5}\right]$$

$$= \frac{1.(2)^{2/3}}{5} \cdot \frac{22}{15} = \frac{22(2^{2/3})}{75}$$

2 (b).



$f(x) = \sin(x)$

$$\text{Vol} = \int_0^{\pi/4} 2\pi x \, f(x)\, dx$$

$$= 2\pi \int_0^{\pi/4} \underbrace{x}_{u} \underbrace{\sin(x)dx}_{dv}$$

so let $u = x \Rightarrow du = dx$

$\frac{dv}{dx} = \sin(x) \Rightarrow v = -\cos(x)$

$$\int_0^{\pi/4} x\sin(x)dx = uv\Big|_0^{\pi/4} - \int_0^{\pi/4} v\, du$$

$$= -x\cos(x)\Big|_0^{\pi/4} - \int_0^{\pi/4}(-\cos(x))dx$$

$$= \left(-\frac{\pi}{4}\cos\frac{\pi}{4}\right) - \left(-0\cos(0)\right) + \int_0^{\pi/4}\cos(x)dx$$

$$= -\frac{\pi}{4}\frac{\sqrt{2}}{2} + 0 + \sin(x)\Big|_0^{\pi/4}$$

$$= -\frac{\pi\sqrt{2}}{8} + \sin\left(\frac{\pi}{4}\right) - \sin(0)$$

$$= \frac{\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{8}$$

$$\therefore \text{Vol} = \frac{2\pi\sqrt{2}}{\cancel{2}}\left[1 - \frac{\pi}{4}\right] = \pi(4-\pi)\sqrt{2}/4$$

(c) $\int_0^1 x^2 e^x dx = uv\Big|_0^1 - \int_0^1 v\,du$ where $u = x^2,\ dv = e^x dx \Rightarrow v = e^x$

$= x^2e^x\Big|_0^1 - \int_0^1 2xe^x dx$

$= e - 0 - 2\int_0^1 xe^x$. now let $u = x,\ dv = e^x dx$

$= e - 2\left[xe^x\Big|_0^1 - \int_0^1 e^x dx\right] = e - 2\left[e - (e-1)\right]$

$= e - 2.$