Part B (2008) Test 1

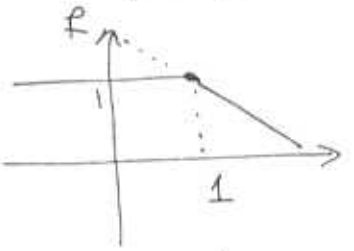
1(a) By "f is continuous at $x=a$" we mean $f(x)$ is defined on some open interval containing the point a and $\lim_{x\to a} f(x) = f(a)$, so the limit on the right exists and has the given value.

By "f is differentiable at $x=a$" we mean the limit

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exists as a real number (which we call $f'(a)$).

Ex. $$f(x) = \begin{cases} 1, & x \le 1 \\ 2-x, & x > 1 \end{cases}$$



then f is continuous at $x=1$, but not differentiable at that point, since $f'_+(1) = -1 \neq 0 = f'_-(1)$.

(b) If $f(x) = \begin{cases} x^2 & x \le 1 \\ ax+b & x > 1 \end{cases}$ $\quad (\Rightarrow f(1) = 1^2 = 1)$

then $\lim_{x\to 1-} f(x) = \lim_{x\to 1-} x^2 = \lim_{x\to 1} x^2 = 1^2 = 1$ &

$\lim_{x\to 1+} f(x) = \lim_{x\to 1} (ax+b) = a+b$ so to be continuous at $x=1$, we must have $\boxed{1 = a+b}$

Also $$f'_-(1) = \lim_{h\to 0-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0-} \frac{(1+h)^2-1}{h}$$

$$= \lim_{h\to 0-} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0-} (2+h) = 2$$

& $$f'_+(1) = \lim_{h\to 0+} \frac{a(1+h)+b-(a.1+b)}{h} = a$$

so to be diff. at $x=1$ we must have $f'_+(1) = f'_-(1) \Rightarrow \boxed{a=2}$

so $\boxed{b=-1}$

$\therefore \quad b = 1-a = 1-2 = -1$

2 (a) $$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

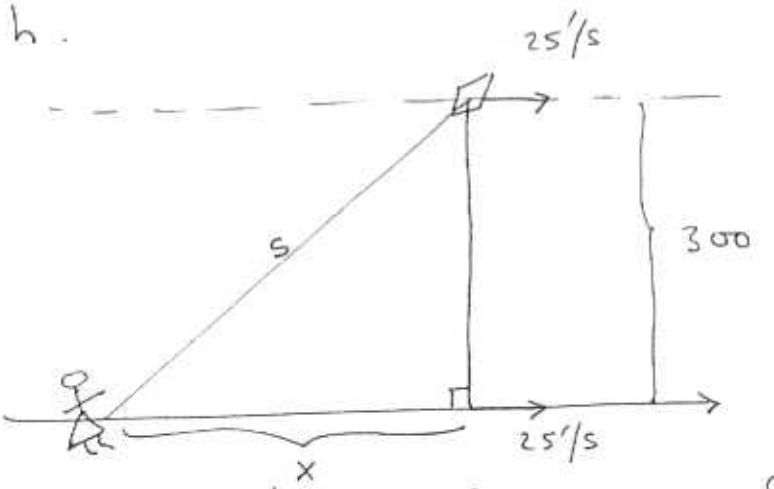
$$= \lim_{h\to 0} \frac{1}{h}\left[(a+h)^3 + 2(a+h)^2 + 1 - (a^3 + 2a^2 + 1)\right]$$, so expanding,

$$= \lim_{h\to 0} \frac{1}{h}\left[\cancel{a^3} + 3a^2h + 3ah^2 + h^3 + \cancel{2a^2} + 4ah + 2h^2 + \cancel{1} - \cancel{a^3} - \cancel{2a^2} - \cancel{1}\right]$$

$$= \lim_{h\to 0} \frac{h(3a^2+4a) + 2h^2 + h^3}{h} = \lim_{h\to 0}\left(3a^2 + 4a + 2h + h^2\right)$$

$= 3a^2 + 4a + 0 + 0$ where we have used the limit theorem "the limit of a sum = sum of the limits" and $\lim_{h\to 0}(2h+h^2) = 2.0 + 0^2 = 0$ since $2h+h^2$ is a continuous function of h.

(b)



Drop a perpendicular from the kite to the ground and let x be the distance from the girl to where the perpendicular meets the ground. Let s be the length of the string. Assume the ground is perfectly flat and the string is perfectly straight! By Pythagoras,

$$s^2 = x^2 + 300^2$$ so by implicit diff.

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 0$$

Given $\frac{dx}{dt} = 25$ this shows $\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt} = 25\frac{x}{s}$.

When the kite is 500' away, $s = 500$ so $500^2 = x^2 + 300^2 \Rightarrow x = 400'$

Hence, at that instant, $\frac{ds}{dt} = \frac{400}{500}.25 = 20'/s$ //