Waikato Scholarship Days
Calculus Workshop Problems

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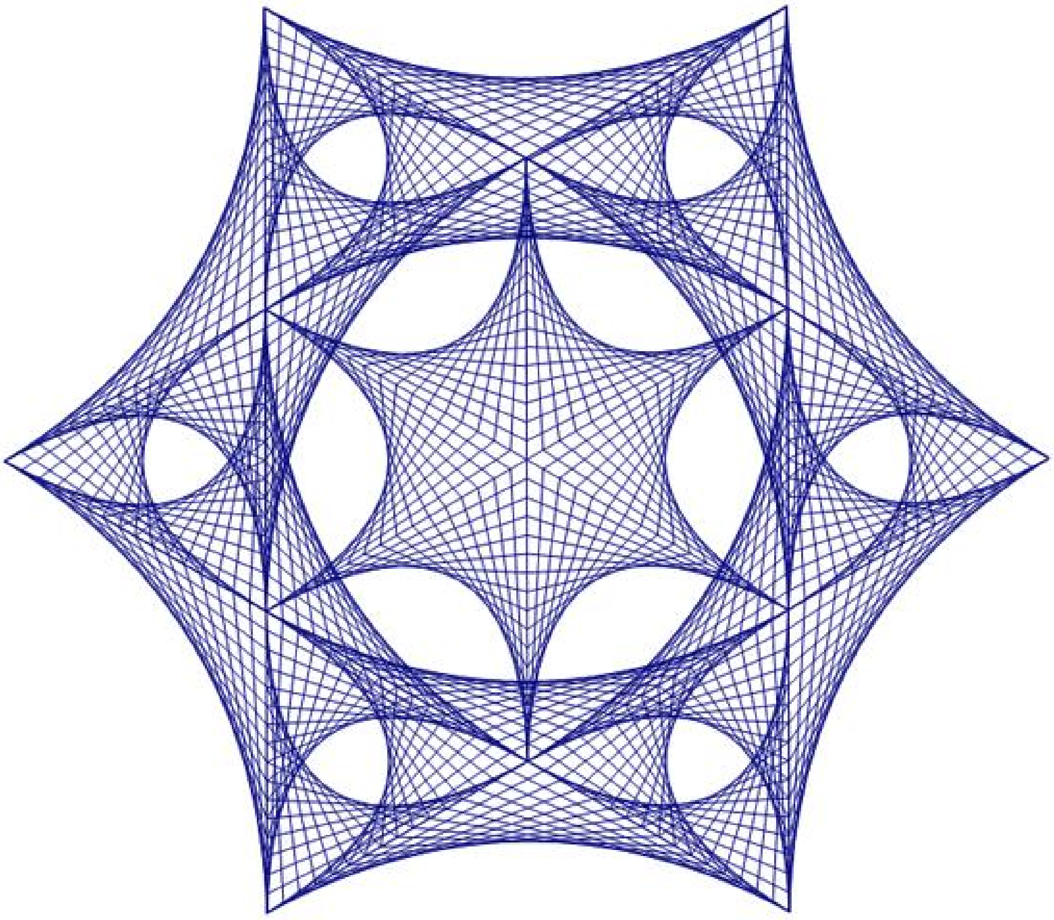

The curves in this striking image are quadratic Bezier curves. These curves are often used in string art. You might like to think about how you would find the equation of such a curve.

The Waikato Scholarship days calculus workshops started in 2006 as a joint initiative by the Waikato Mathematics association and the University of Waikato. Apart from a hiatus during covid the workshops have run every year since then.

This edition was created for the 2023 workshop convened by

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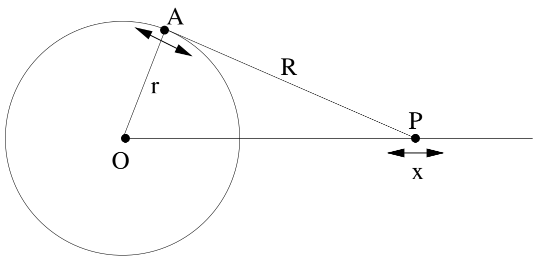
If you have a question about any of these problems feel free to contact us.

Problems

How Far Apart (⋆ ⋆)

What is the minimum distance between the two curves $y = a^x$ and $y = \log_a(x)$.

Piston (⋆ ⋆ ⋆)



A rotating crankshaft with radius r centre O and endpoint A is connected to a piston arm of length R. The piston at P moves back and forth along the x-axis so that its position at time t is x. If the piston is fully extended at time $t = 0$ and the crankshaft rotates at a constant ω radians per second, give an equation for x as a function of t. Hence show that when the angle at A is a right angle, the speed of the piston is

$$\frac{dx}{dt} = -\frac{\omega r^2 \sqrt{R^2 + r^2}}{R^2 + r^2 - rR}$$

Triples (⋆)

If the three numbers a, b and c satisfy the equations $3a+4b = 3c$ and $4a-3b = 4c$ show that $a^2 + b^2 = c^2$.

Gompertz (⋆ ⋆)

The Gompertz growth function is sometimes used to describe bounded growth in biology. It takes the form

$$G(t) = ke^{(-Ae^{-Bt})}$$

where A, B and k are positive constants. Show that the function $G(t)$ is always increasing and find $\lim_{t\to\infty} G(t)$. What is the maximum rate of growth of $G(t)$ and at what time does this maximum occur.

Exactly! (⋆ ⋆ ⋆)

Find an exact expression involving surds for

(a) $\cos(75°)$ **(b)** $\tan(15°)$ **(c)** $\sin(22.5°)$ **(d)** $\sin(72°)$

hint: The problem "Diabolical", *might help with part (d).*

Ovalsize (⋆ ⋆)

The area of the ellipse with equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is πab. In this question we want you to prove this formula

(a) Set up an integral to describe this area.

(b) Substitute $x = a\sin(t)$ in your integral.

(c) Integrate and solve.

Minillipse (⋆ ⋆)

What is the smallest area of an ellipse containing the rectangle with sides A and B.

See the problem "Ovalsize" *for the area formula.*

Ellipses (⋆ ⋆)

Two ellipses of the same shape, one oriented horizontally and one vertically, cross at the four points $(x, y) = (\pm 1, \pm 1)$ where they meet at an angle of $60°$. What is the area of each ellipse?

See the problem "Ovalsize" *for the area formula.*

Logarhythmia (⋆)

If a and b are two numbers so that $a^2 + b^2 = 7ab$, show that

$$\log\left(\frac{a+b}{3}\right) = \frac{1}{2}\Big(\log(a) + \log(b)\Big)$$

K-I-S-S-I-N-G (⋆ ⋆) Ellipse and Hyperbola in a tree, K-I-S-S-I-N-G.

Give a condition on the values a and b so that the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is tangent to the hyperbola $xy = 1$.

Twice Tangent (⋆ ⋆ ⋆)

Consider the two parabolas with equations $y = x^2$ and $y = x^2 + 2x + 2$. There is a unique straight line which is tangent to both curves. Find the equation of this line.

Twice Tangent II? (⋆ ⋆ ⋆ ⋆)

Find the two points on the curve $y = x^4 - 2x^2 - x$ which have a common tangent line.

Flip It (⋆) Find the area between the curves $3y^2 = 4 - x$ and $y^3 = x$.

Palindromic Quartics (⋆ ⋆)

A polynomial is called **palindromic** if the coefficients make a palindrome. A general palindromic quartic equation has the form

$$ax^4 + bx^3 + cx^2 + bx + a = 0$$

Note that $a \neq 0$ else the equation wouldn't be quartic, hence $x = 0$ is not a root. Solving a general quartic equation is difficult, but if the quartic is palindromic there is a trick we can use. This question is about that trick.

Dividing the equation by x^2 will not change the solution since we know $x = 0$ is not a root. This gives the equation

$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$$

Substituting $y = x + \frac{1}{x}$ allows us to rewrite this as a quadratic equation (try it) which we can solve to find y. Once we know y we can use it to find x.

Use this method to find all four roots of the palindromic quartic equation

$$12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$$

Simultaneous Conics (⋆ ⋆)

Solve the system of equations

$$\begin{aligned} x^2 + 2xy + 3y^2 &= 2 \\ 3x^2 - 5xy + 7y^2 &= 15 \end{aligned}$$

Try the following

(a) Combine the two equations to get an equation with constant term 0.

(b) Factorise this to find y in terms of x.

(c) Substitute into one of the original equations to get a quadratic.

(d) Solve to find x. Substitute to find y.

Complex Trig Formulae (⋆ ⋆ ⋆)

Let $z = \cos(\theta) + i\sin(\theta)$ so that $z^{-1} = \overline{z} = \cos(\theta) - i\sin(\theta)$. De Moivre's theorem then gives

$$z^n + z^{-n} = 2\cos(n\theta) \qquad z^n - z^{-n} = 2i\sin(n\theta)$$

These formulae can help us solve trig identities. Rewriting everything in terms of z can turn a tricky trig identity into a not quite so tricky polynomial identity.

Use this method to prove the following trig identities

(a) $32\cos^6(\theta) = \cos(6\theta) + 6\cos(4\theta) + 15\cos(2\theta) + 10$

(b) $128\cos^3(\theta)\sin^5(\theta) = \sin(8\theta) - 2\sin(6\theta) - 2\sin(4\theta) + 6\sin(2\theta)$

Systematic (⋆ ⋆)

Solve this system of equations for $x, y \in \mathbb{C}$.

$$\begin{cases} (2x-y)^2 - 4(2x-y) &= 5 \\ x^2 - y^2 &= 3 \end{cases}$$

AB-solutely (⋆ ⋆ ⋆)

A line AB of constant length d moves with the end point A always on the x-axis and with B always on the line $y = 6x$. Find the locus of the midpoint.

Asymptotics (⋆ ⋆ ⋆)

Asymptotes and Hyperbola Equations

Consider two lines with equations $ax + by = p$ and $cx + dy = q$. Then the equation for any hyperbola with these lines as asymptotes can be written in the form

$$(ax + by - p)(cx + dx - q) = k$$

for some constant k.

(a) Find the equation of the hyperbola with asymptotes $x + y = 1$ and $2x - y = 3$ which passes through the point $(x, y) = (1, 2)$.

(b) Graph the hyperbola $(2x - y - 1)(2y - x - 1) = 4$. Find the center, the equation of the major axis, and the vertices.

Basic Parts (⋆ ⋆)

Integration by Parts Formula

There is no product rule as such for integration. However integrating the product rule for derivatives gives the following useful formula known as **Integration by Parts**.

$$\int u\,dv = uv - \int v\,du$$

For example consider the integral $\int xe^x\,dx$. We choose $u = x$ and $dv = e^x\,dx$ so that this integral is $\int u\,dv$. To use the formula we need to know du and v. But since $\frac{du}{dx} = 1$ we have $du = dx$; and integrating $\int dv = \int e^x\,dx$ gives $v = e^x$. Plugging into the integration by parts formula gives

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + c$$

which can be directly verified by differentiation.

Use Integration by Parts to find the following

(a) $\int x\cos(x)\,dx$.

(b) $\int x^2e^x\,dx$.

(c) $\int \ln(x)\,dx$.

An e-sy one (⋆) Determine $\displaystyle\int_e^{e^2} \frac{dx}{x(\ln x)^3}$

Midlog (⋆)

Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be two distinct points on the graph of $y = \log(x)$. A horizontal line through the midpoint of $\overline{AB}$ meets the graph $y = \log(x)$ at the point $C = (x_3, y_3)$. Show that $x_3 = \sqrt{x_1x_2}$.

Concentric (⋆ ⋆) Find the equation of the locus of the points of intersection of any two perpendicular tangents to the circle $x^2 + y^2 = r^2$.

Between (⋆ ⋆)

$$\begin{aligned} y &= (x+1)^2 - a \\ y &= a - (x-1)^2 \end{aligned}$$

Find the value of a so that the curves above bound an area of 1 square unit.

Integrating factors (⋆ ⋆ ⋆)

(a) Show that the differential equation $\dfrac{dy}{dx} + ky = f(x)$ can be written in the form $\dfrac{d}{dx}\left(ye^{kx}\right) = e^{kx} f(x)$.

Use this trick to solve the differential equation

$$\frac{dy}{dx} + ky = e^x$$

(b) A particle is subject to several forces so that its velocity $v(t)$ changes according to the differential equation

$$v'(t) + kv(t) = (1-2t)e^{-kt^2}$$

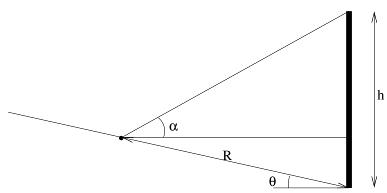
where $k > 0$ is constant.

If the object starts from rest, find v as a function of t.

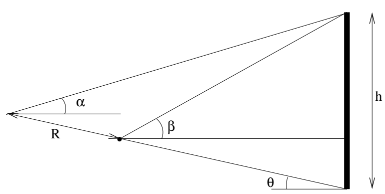
(c) Find the time at which the object changes direction and show that after this time it never stops moving again.

Monolith (⋆)

A martian rover finds a monolith on Mars. It is located on a slope with an angle to the horizontal of θ. At a distance R upslope from the monolith the rover's camera measures an angle of elevation α to the top of the monolith. Find a formula for the height of the monolith in terms of R, θ and α.



Monolith 2 (⋆ ⋆)



A martian rover finds another monolith on Mars. It is located on a slope with an angle to the horizontal of θ. Two angles of elevation α and β to the top of the monolith are measured from points a distance R apart directly upslope from the monolith. Find a formula for the height of the monolith in terms of R, θ, α and β.

Deviation (⋆ ⋆)

The famous normal distribution curve used in statistics has equation

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Find the points of inflection of this curve.

Tea Formulae (⋆ ⋆)

Let $t = \tan\left(\frac{x}{2}\right)$. Find expressions for $\sin(x)$, $\cos(x)$ and $\tan(x)$ in terms of t. Show also that

$$dx = \frac{2\,dt}{1+t^2}$$

Substituting for t is a useful trick when integrating as it will convert integrals involving trigonmetric functions into integrals involving rational functions.

Exact Fract (⋆ ⋆ ⋆) Find the exact value of $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{8+\cos^2 x}\,dx$. Partial fractions may be helpful.

Fibs (⋆ ⋆)

Fibonacci numbers are the famous sequence

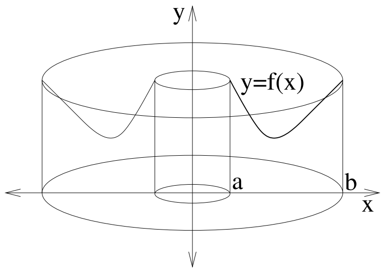
$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

defined by the mathematician Leonardo of Pisa in 1202. The numbers F_n in this sequence satisfy the Fibonacci recursion relation $F_{n+1} = F_n + F_{n-1}$ which states that each new number is the sum of the two that precede it. To start the sequence we must know the first two numbers. The Fibonaccis sequence results by choosing $F_0 = 0$ and $F_1 = 1$. Other sequences can be formed by starting with two different numbers.

A geometric sequence is a sequence where the ratio between adjacent terms in the sequence is constant. Find all geometric sequences that satisfy the Fibonacci recursion relation. If the first numbers for a sequence satisfying Fibonacci recursion are $G_0 = a$ and $G_1 = b$ then give a condition on a and b that will ensure that the sequence is geometric.

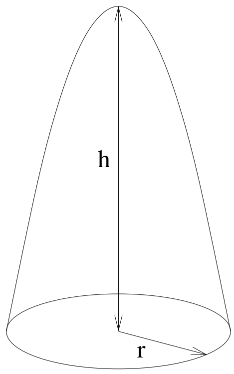
Shells (⋆ ⋆)

Volume of Revolution : Shell Method



If $0 \le a \le b$ then the volume obtained by rotating the area under the curve $y = f(x)$ between $x = a$ and $x = b$ around the y-axis is

$$\text{Volume} = 2\pi \int_a^b xy\, dx = 2\pi \int_a^b x f(x)\, dx$$



(a) Apply the shell method to the constant function $y = h$ between $x = 0$ and $x = r$. Verify that it gives the expected formula for the volume of a cylinder.

(b) Apply the shell method to $y = h\left(1 - \frac{1}{r}x\right)$ between $x = 0$ and $x = r$ and verify that it gives the expected formula for the volume of a cone.

(c) Use the shell method to find the volume of a paraboloid of height h and base radius r.

Think First (⋆)

Minimise the function $y = e^{\left(5 - \sqrt{\frac{1}{x^2+2x+2}}\right)}$

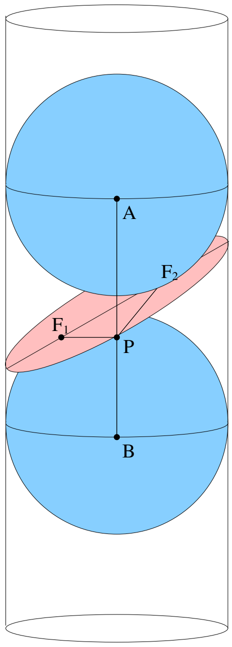
The Formula Applies (⋆ ⋆ ⋆)

Find all complex roots of the equation $z^4 + 2iz^2 + 1 = 0$. Sketch these roots on an Argand diagram. The roots make the corners of a polygon. Find the area of this polygon.

Greek Geometry (⋆ ⋆)

The ancient Greek mathematicians didn't have algebra or calculus. But they could do some pretty clever things with straight geometry. Reason like an ancient Greek to solve this problem.

A vertical cylinder is cut by a plane. Spheres with the same radius as the cylinder sit inside the cylinder above and below the plane and tangent to it at points F_1 and F_2. Suppose P is a point on the intersection of the plane and the cylinder. A vertical line through P is tangent to the spheres at points A and B.



(a) Explain why $|PF_1| = |PB|$ and $|PF_2| = |PA|$.

(b) Hence show that $|PF_1|+|PF_2|$ is the same for all points P on the intersection of the cylinder and plane. Conclude that the intersection is an ellipse with foci at F_1 and F_2.

(c) Essentially the same argument also works in a cone, but the two spheres will need to be of different sizes.

This is just Ab-surd (⋆ ⋆)

Solve $\sqrt{x} + \sqrt{x+1} = 3$.

This is just Ab-exp? (⋆ ⋆)

Solve $e^{2x} + e^{x+1} = 1$.

This is just Ab-log? (⋆ ⋆)

Solve $\log_{10}(x+2) + \log_{10}(x-1) = 1$.

This is just Ab-tan? (⋆ ⋆)

Solve $\tan^{-1}(x) + \tan^{-1}(2x) = \dfrac{\pi}{4}$ where $\tan^{-1}$ is the inverse tangent.

This is just Ab-???? (⋆ ⋆)

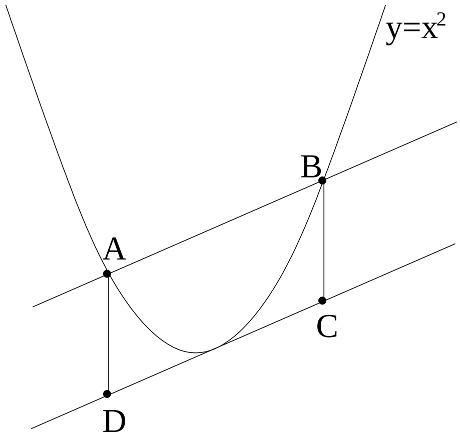
Solve $2\sin^{-1}(x) + \cos^{-1}(x) = \pi$ where $\sin^{-1}$ and $\cos^{-1}$ are inverse trig functions.

Max Co (⋆ ⋆)

Find the largest coefficient in the binomial expansion of $(3x + 2)^{100}$.

Hint: When is a_{n+1} bigger than a_n?

Archimedes (⋆ ⋆)



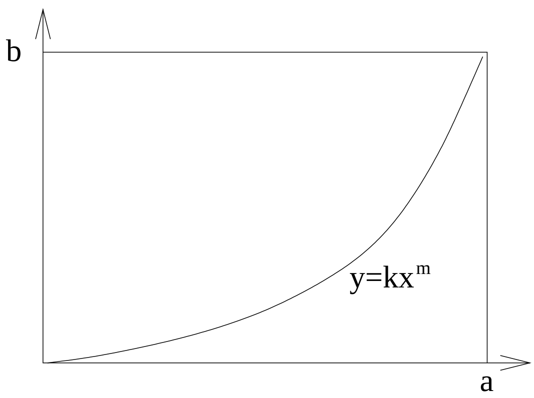
Let A and B be two points on the parabola $y = x^2$. Let $ABCD$ be a parallelogram with AD vertical and CD tangent to the parabola.

Show that the part of the parallelogram above the parabola is exactly twice as large as the part below it.

Cauchy Schwarz (⋆ ⋆)

Show that $(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)$ for all $a, b, c, d \in \mathbb{R}$.

Depend on it (⋆)



A rectangle with sides a and b is drawn around the function $y = kx^m$ in the positive quadrant as shown. Prove that the fraction of the rectangle under the curve depends only on m and not on the size of the rectangle.

Discrimination (⋆ ⋆ ⋆)

A quadratic polynomial $y = ax^2+bx+c$ has two distinct real roots when $b^2-4ac > 0$. Discuss how you could test whether or not a cubic polynomial $y = ax^3+bx^2+cx+d$ has three distinct real roots.

Inverse Trig ($\star\star\star$)

Inverse Trig Functions

The inverse sine function $\sin^{-1}(x)$ is the function corresponding to the button $\sin^{-1}$ on most calculators. It gives the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x. Hence

$$\begin{aligned} \sin^{-1}(\sin(\theta)) &= \theta \qquad \text{for all } -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2} \\ \sin(\sin^{-1}(x)) &= x \qquad \text{for all } -1 \leq x \leq 1 \end{aligned}$$

The functions $\cos^{-1}(x)$ and $\tan^{-1}(x)$ are similarly defined.

Note that in this notation the negative one is NOT a power; it just means inverse function. This is inconsistent with notation like $\sin^2(x)$ where the 2 is a power. We agree that this clash of notation sucks but we seem to be stuck with it for historical reasons.

(a) Show that $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$ for $-1 \leq x \leq 1$.

(b) Show that $\sin^{-1}(\cos(\theta)) = \frac{\pi}{2} - \theta$ for $0 \leq \theta \leq \pi$.

(c) Find the derivative of $\sin^{-1}(x)$ as a function of x.

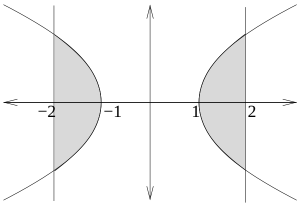
Ellipse Meet Hyperbola ($\star\star\star$)

Suppose that ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ crosses hyperbola $\dfrac{x^2}{c^2} - \dfrac{y^2}{d^2} = 1$ at right angles.

Show that $a^2 - b^2 = c^2 + d^2$.

Just a Simple Area? (⋆ ⋆ ⋆)

(a) Give a formula involving an integral for the total area between the hyperbola $x^2 - y^2 = 1$ and the vertical lines at $x = \pm 2$.



(b) A table of integrals gives

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left| x + \sqrt{x^2 + a^2} \right| + c$$

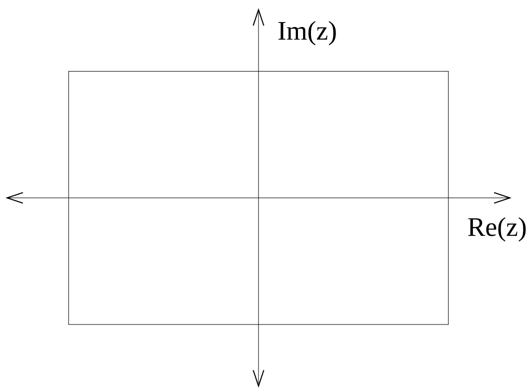
Verify this by taking the derivative.

(c) Use the integral formula in part (b) to find the area in part (a).

Rectangularity (⋆ ⋆)

Let $p(x)$ be a quartic (degree 4) polynomial with real coefficients. Suppose that the roots of this polynomial in the complex plane lie on the corners of a rectangle centered at the origin with sides parallel to the real and imaginary axes.

Show that $p(x)$ has no terms involving odd powers of x.



Reflect on This (⋆)

Find the coordinates of the point obtained by reflecting $(1, 3)$ in the line $y = 2x + 11$.

Yet Another Cubic Conundrum (⋆)

Let $p(x)$ be a cubic polynomial with real coefficients. Suppose that its derivative $p'(x)$ has complex roots $a \pm bi$. Show that $p(x)$ has a point of inflection at $x = a$.

Totally Rooted ($\star\star\star$)

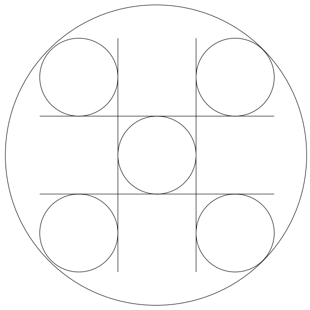
Consider the set $\{\omega_1, \omega_2, \cdots, \omega_n\}$ of roots of the polynomial $z^n - 1$.

(a) Show that $\omega_1 + \omega_2 + \cdots + \omega_n = 0$.

(b) Show $\omega_1\omega_2 \cdots \omega_n = (-1)^{n+1}$.

(c) Show that the sum of all products $\omega_i\omega_j$ of two different roots is zero. Ordering the roots for convenience with $i < j$ we write this as

$$\sum_{i<j} \omega_i\omega_j = 0$$



Quincunx ($\star$)

In the figure shown to the left (which is called a quincunx) what is the ratio of the radius of the larger circle to that of the smaller ones?

Not too hard ($\star$)

Find the area between the curve $y^2 = 2y + x$ and the line $x - y = 1$.

Distance ($\star\star\star$)

What is the distance between the curve $y^2 - 4y - x + 5 = 0$ and the origin.

Reticulating Splines ($\star\star$)

A modeller wishes to construct a piecewise function $f(x)$ on the interval $0 \le x \le 1$ using two parabolic pieces joined at $x = a$, where $0 < a < 1$.

The first piece, which is used to give the value when $0 \le x \le a$, is a parabola with minimum at the origin. The second piece, which is used to give the value when $a < x \le 1$, is a parabola with maximum at the point $(1, 1)$.

For each value of a there is a unique function of this type which is differentiable in the region $0 \le x \le 1$. Find this function.

Hyperbolic (⋆ ⋆ ⋆)

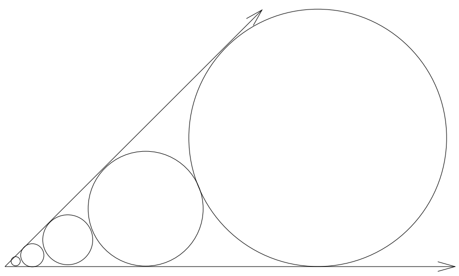
The hyperbola with equation $11x^2 + 24xy + 4y^2 = 500$ is centered on the origin and symmetrical about the line $3x - 4y = 0$. Rewrite the equation using rotated coordinates $u = \frac{4}{5}x + \frac{3}{5}y$ and $v = \frac{3}{5}x - \frac{4}{5}y$. Find the vertices, the foci and the equations of the asymptotes. Express all answers using x and y coordinates.

Triple Pointer (⋆ ⋆)

A quadratic function $y = q(x)$ is uniquely determined by three of its values. Find the quadratic function with $q(1) = 4$, $q(2) = 1$ and $q(3) = 2$.

Diminishing Circles (⋆ ⋆ ⋆)

A circle of radius 1 is drawn tangent to the x axis and to the line $y = x$. A sequence of circles is then added with each new circle being tangent to the previous one and to the two lines as indicated in the diagram. Find the ratio of the areas of two adjacent circles in this sequence. Hence or otherwise find the total area enclosed by the circles.



Min to Max (⋆ ⋆ ⋆)

A quartic polynomial passes through the origin and has local minima at $(1, -1)$ and at $(-1, -4)$. Find the coordinates of the local maximum.

A Systematic Cubic (⋆)

A cubic polynomial has a local minimum at $(-1, -20)$ and a point of inflection at $(-2, -18)$. Find the y intercept.

Tangentially (⋆)

There are two lines which pass through the point $(1, 1)$ and are tangent to the curve $y = x^2 + 5x + 6$. Find their equations.

Objectivity ($\star\star$)

Consider the region R bounded by the two curves $y = 2 - x^2$ and $y = \frac{1}{x}$. We are interested in finding the maximum and minimum values of the linear objective function $P(x, y) = ax + by$ on R.

(a) Find where these curves intersect and graph the region R.

(b) Give a necessary and sufficient condition on a and b so that both the maximum and minimum values of $P(x, y)$ do not occur at points where the two curves intersect.

The Fundamental Problem ($\star\star\star$)

State the Fundamental Theorem of Calculus and use examples to demonstrate what it tells us about derivatives and integrals.

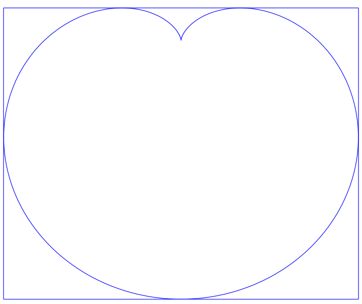
Piecewise ($\star$)

Consider the function $f(x) = \begin{cases} ax^2 + bx + c & , \quad x < 1 \\ 2 & , \quad x = 1 \\ dx + e & , \quad x > 1 \end{cases}$

Suppose $f(x)$ is differentiable with a maximum at $x = 0$ and an x-intercept at 3. Find the constants a, b, c, d, e.

Valentine ($\star\star$)

A mathematically oriented confectionary company makes Valentine's Day chocolate in the shape of a cardioid with equation $r = 1 - \sin(\theta)$ (dimensions are centimetres). Each chocolate is individually wrapped in red foil and shipped in a box just big enough to hold it. Find the dimensions of the rectangular base of the box.



Circulation ($\star\star$)

Find values of a, b and c so that the circle with equation $x^2 + y^2 = ax + by + c$ passes through the three points $(-1, 2)$, $(1, 3)$, and $(2, -1)$. Hence or otherwise find the coordinates of the center and the radius of this circle.

L'Hôpital Rules (⋆ ⋆ ⋆)

L'Hôpital's Rule

L'Hôpital's Rule is a powerful technique for computing limits of fractions in the case where the top and bottom of the fraction both go to zero or both go to infinity. If this happens we say that the limit is **indeterminate** and L'Hôpital's rule then tells us that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Note that the limit must be indeterminate for L'Hôpital's rule to work. If you try to apply L'Hôpital's rule to a limit which is NOT indeterminate it will give a wrong answer.

Use L'Hôpital's rule to compute

(a) $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$

(b) $\displaystyle\lim_{x\to 1}\frac{\ln(x)}{\sin(\pi x)}$

(c) $\displaystyle\lim_{x\to 1}\frac{1+\cos \pi x}{x^2-2x+1}$

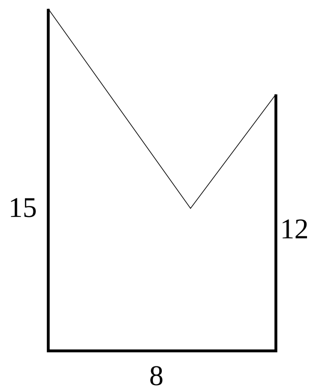
(d) $\displaystyle\lim_{x\to 0}\frac{5^x-3^x}{x}$

(e) $\displaystyle\lim_{x\to \frac{\pi}{2}}\frac{\sec(x)}{1+\tan(x)}$

(f) $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x^3}$

Hanging Around (⋆ ⋆ ⋆)

Two poles, one with a height of 15 metres the other with a height of 12 metres are 8 metres apart. A rope with a length of 10 metres connects the tops of the two poles. A weight is placed on the rope so that it is free to slide along it. How far above the ground does the weight hang? [Assume that when the rope is pulled taut it makes a straight line]

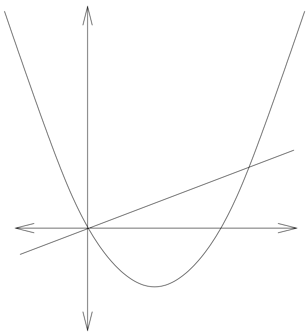


Direct Explanations (⋆ ⋆)

Explain how a parabola can be described using a focus and a directrix. Find the focus and directrix for the parabola $y = x^2 - 5x + 6$.

Par Secs (⋆ ⋆ ⋆)

A line normal to a parabola will (if not parallel to the axis) cut the parabola at a second point and will bound an area with the parabola known as a **parabolic sector**.



(a) If $k > 0$. Find the equation of the line normal to parabola $y = x^2 - kx$ at the origin and find the coordinates (a, b) of the other point where this line meets the parabola.

(b) Find the area A of the corresponding parabolic sector and show that A will be minimum when a is minimum.

(c) What is the minimum area of a parabolic sector for the parabola $y = x^2$.

Near the Edge (⋆ ⋆)

Consider the region consisting of the points in the interior of a square which are closer to the center of the square than they are to the edge. If the square has side length a, then what is the area of this region?

Triple Circle (⋆ ⋆)

You are given an equilateral triangle whose sides have length 1 and a circle of radius r centered at the centre of the triangle. What value of r will minimize the area of the region consisting of those points which either lie inside the circle but outside the triangle or inside the triangle but outside the circle?

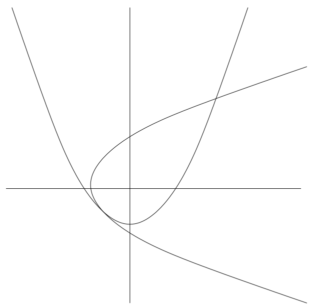
What if the equilateral tiangle is replaced by a regular n -gon?

Parabolabola (⋆ ⋆ ⋆)

For which values of a are the parabolas
$y = x^2 + a$ and $x = y^2 + a$ tangent?

For one value of a, the interiors of the parabolas will be disjoint, while for the other value (the case shown in the diagram) the interiors will overlap.

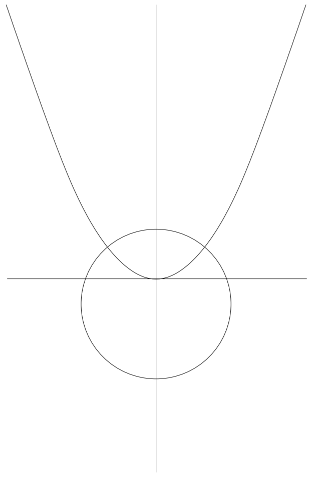
In this second case find the area of the region of overlap.



Par for the Circle (⋆ ⋆)

A circle of radius 1 whose center is on the y-axis is normal to the parabola $y = x^2$ as shown in the figure.

Find the y-coordinate of the center of the circle.



The Tenth degree (⋆ ⋆ ⋆)

Find all solutions to the equation $\cos^{10}(x) - \sin^{10}(x) = 1$ in the interval $0 \leq x \leq 2\pi$.

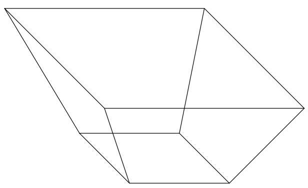
be-cos be-cos be-cos (⋆ ⋆)

By multiplying both sides by $8\sin(b)$ or otherwise, solve the equation

$$\cos(b)\cos(2b)\cos(4b) = \frac{1}{8}$$

Log Log (⋆ ⋆)

For what value of a is the curve $y = a^x$ tangent to the line $y = x$.



Its a Trap! (⋆ ⋆ ⋆)

You want to make a planter in the shape of an open box with a square base and four congruent trapezoidal sides. If the total surface area of the base and sides is 4 square metres, find the dimensions that will maximise the volume of the planter?

Efficient Coefficients (⋆ ⋆)

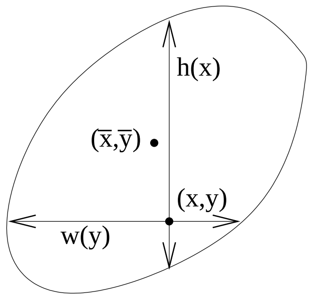
(a) For what values of the constant $a > 0$ is the coefficient of x^3 in the expansion of $(x + a)^{10}$ the largest coefficient?

(b) Which values of $a > 0$ make the coefficient of x^{43} the largest coefficient in the expansion of $(x + a)^{100}$.

Centroid (⋆ ⋆ ⋆)

The centroid of a 2-D shape

The centroid of a 2-D shape is the point about which the moment of the shape is zero. A 2-D shape of uniform density would balance on this point. Sometimes this point is called the center of mass.



If the height of the shape at position x is $h(x)$ and the width of the shape at position y is $w(y)$ then the centroid has coordinates

$$(\overline{x}, \overline{y}) = \left(\frac{1}{A} \int x.h(x)\, dx \ , \ \frac{1}{A} \int y.w(y)\, dy \right)$$

where $A = \int h(x)\, dx = \int w(y)\, dy$ is the area.

(a) Find the centroid of the area between the curve $y = 1 - x^2$ and the x-axis.

(b) Locate the centroid of a half disk (half of a circular disk) of radius r.

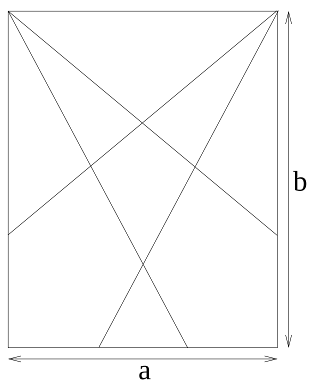
(c) Find the centroid of the triangle with corners at $O = (0, 0)$, $P = (b, 0)$ and $Q = (0, h)$. Show that your answer lies on the medians (lines through a vertex and the center of the opposite side) of the triangle.

Logarithmic differentiation (⋆ ⋆)

Find the derivative of the function $y = \ln(x)^{\left(\frac{1}{\ln(x)}\right)}$ assuming that $x > 1$.

Origami (⋆ ⋆ ⋆ ⋆)

A rectangular piece of paper of width a and height b is divided into three pieces of equal area by lines drawn through one of the corners. It is then similarly divided by lines drawn through another corner. What is the area of the kite shaped piece in the center?



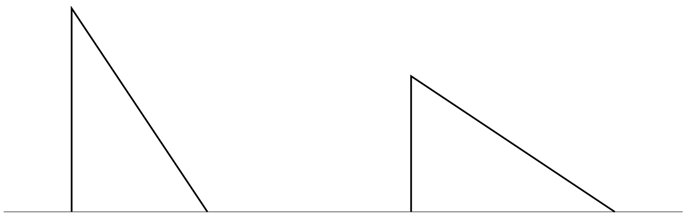
It all starts with V (⋆ ⋆)

If $y = vx$ where v is a function of x, how are $\dfrac{dy}{dx}$ and $\dfrac{dv}{dx}$ related? Hence or otherwise solve the differential equation

$$x\frac{dy}{dx} + 3x + 3y = 0$$

Give me a Break (⋆)

A vertical pole breaks but does not come apart. The top falls over and touches the ground a distance 6 metres away from the base. The pole is then repaired, but later breaks again at a point 3 metres lower on the pole. This time the base touches the ground 12 metres from the base. How tall is the pole when not broken?



Meet (⋆ ⋆) Find both coordinates of all the points where the circle with center at $(1, -1)$ radius 3 intersects the circle with center at $(-2, 0)$ radius 2.

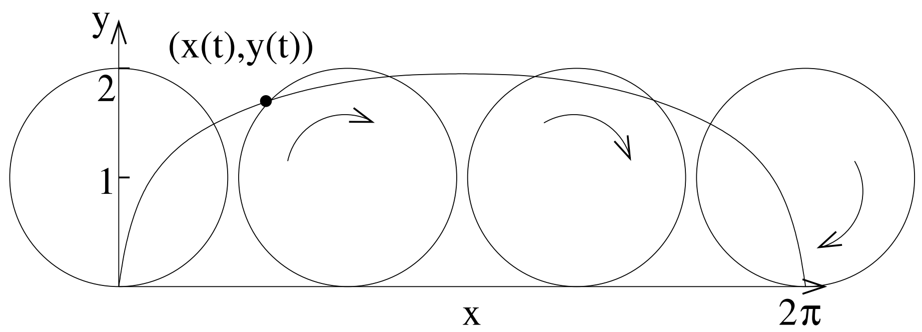
Quasillipse (⋆)

Consider the points $X = (-a, 0)$ and $Y = (a, 0)$. Let k be a constant and let $P = (x, y)$. The points P with $|PX| + |PY| = k$ form an ellipse provided $k > 2a$. The points P with $|PX| - |PY| = k$ give a branch of a hyperbola.

(a) What shape do you get by requiring that $|PX|^2 + |PY|^2 = k$?

(b) What shape do you get by requiring that $|PX|^2 - |PY|^2 = k$?

Cycloid (⋆ ⋆ ⋆) This question concerns a **cycloid**, the curve traced out by a point on the rim of a wheel as it rolls along the ground.



Speed and Distance in 2-D

If an object moves in the plane so that its coordinates at time t are $(x(t), y(t))$, then its speed at time t is given by the equation

$$\text{speed}(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

and the distance that it travels between $t = a$ and $t = b$ is

$$\text{distance} = \int_a^b \text{speed}(t)\, dt$$

(a) A wheel of radius 1 rolls at a constant rate of one metre per second. Using a coordinate system where the 'ground' is along the x-axis and the center of the circle moves from $(0, 1)$ to $(2\pi, 1)$ during one revolution, give coordinates $(x(t), y(t))$ at time t for the point on the rim of the circle which was in contact with the ground at time $t = 0$.

(b) Hence or otherwise compute the length of the cycloid, the total distance travelled by a point on the rim during one revolution of the wheel.

Locus Pocus ($\star\star$)

Let P and Q be two points in the plane a distance d apart. Let $a > 0$ be constant. If X is an arbitrary point in the plane, let $|XP|$ and $|XQ|$ denote the distances from X to P and from X to Q respectively. Show that the locus of points in the plane where $|XP| = a|XQ|$ is a circle; describe the location of its centre; and give a formula for the radius of this circle in terms of d and a.

The Doomsday Equation ($\star\star$)

Let $c > 0$ and $k > 0$ be constants. The differential equation $\dfrac{dy}{dt} = ky^c$ is called a Doomsday equation if $c > 1$.

(a) Solve this differential equation.

(b) Show that there is a finite time $t = T$ (Doomsday!) so that $y \to \infty$ when $t \to T$.

(c) The population growth curve of an especially prolific breed of rabbit satisfies this equation with $c = 1.01$. If initially there are two rabbits, and there are 16 rabbits after 3 months, when is doomsday.

Integration for Experts ($\star\star$)

(a) Find $\displaystyle\int \cos^2(x)\,dx$. A trig identity might be useful.

(b) Substituting $\sin(t) = x$ simplifies $\displaystyle\int \sqrt{1-x^2}\,dx$. Check it out.

(c) Compute $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$ and confirm for yourself that the area of a circle of radius 1 is π.

(d) Compute $\displaystyle\int_{a}^{1} \sqrt{1-x^2}\,dx$. Hence **or otherwise** find a formula for the area of a segment of a circle.

Geometric ($\star\star$)

The first k terms of a geometric sequence sum to x. Their reciprocals sum to y. Find an expression for the product of the first and kth terms in terms of x and y.

Unreal (⋆ ⋆)

If a and b are distinct complex numbers with modulus 1, show that the complex number $c = \dfrac{a+b}{a-b}$ is imaginary.

The Shortest Distance (⋆ ⋆)

Find a formula for the distance between point (x_0, y_0) and the line $y = mx + b$.

A Systematic Cubic (⋆)

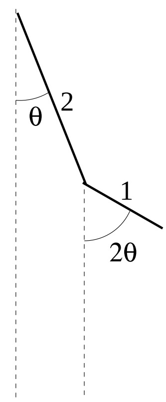
A cubic polynomial has a local minimum at $(-1, -20)$ and a point of inflection at $(-2, -18)$. Find the y intercept.

Basic Training (⋆)

One train travels west towards Chicago at 120kph while another travels north away from Chicago at 90kph. At time $t = 0$ the first train is 10km east and the second train is 20km north of the Chicago station. Calculate the rate at which the distance between them is changing

(a) at time $t = 0$ **(b)** 10 min later.

Epicircularity (⋆ ⋆ ⋆ ⋆)



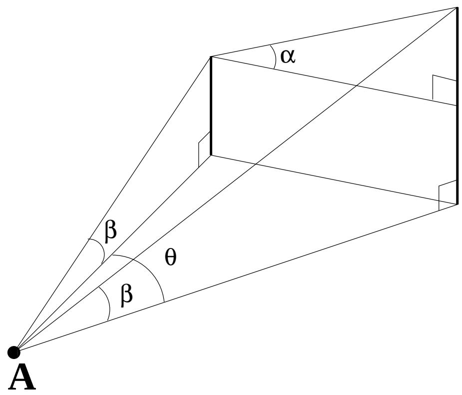
A rigid arm of length 2m is rotated in the vertical plane by a small electric motor at constant angular velocity about a bearing at one end. Attached to the end of this arm is a smaller arm of length 1m which is also rotating about its point of attachment at the same constant angular rate. The two arms start by hanging vertically from their points of attachment so that the minimum height of the tip of the shorter arm is 3m below the point of attachment. As the arms rotate the angle which the smaller arm makes to the vertical is always twice that which the longer arm makes. What is the maximum height attained by the tip of the smaller arm?

Fun with Surveying (⋆ ⋆ ⋆)

I am standing at point A and I can see two towers. The angle of elevation β of the towers from my position is the same. However I know that one tower is actually twice as tall as the other and that when standing at the top of the shorter tower, the angle of elevation to the top of the taller tower is α.

Show that the horizontal bearing θ between the two towers from my position is

$$\theta = \cos^{-1}\left(\frac{5\cot^2\beta - \cot^2\alpha}{4\cot^2\beta}\right)$$



Integrating factors (⋆ ⋆ ⋆)

(a) Show that the differential equation $\frac{dy}{dx} + ky = f(x)$ can be written in the form $\frac{d}{dx}\left(ye^{kx}\right) = e^{kx}f(x)$.

Use this trick to solve the differential equation

$$\frac{dy}{dx} + ky = e^x$$

(b) A particle is subject to several forces so that its velocity $v(t)$ changes according to the differential equation

$$v'(t) + kv(t) = (1-2t)e^{-kt^2}$$

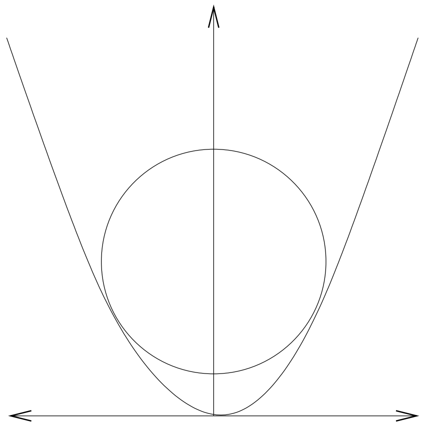
where $k > 0$ is constant.

If the object starts from rest, find v as a function of t.

(c) Find the time at which the object changes direction and show that after this time it never stops moving again.

Simply Natural (⋆ ⋆)

If $x = y\ln(xy)$, prove that $\frac{dy}{dx} = \frac{y(x-y)}{x(x+y)}$.



Hocus Focus (⋆ ⋆)

A circle of radius 1 is tangent to the parabola $y = x^2$ as shown.

Find the center of the circle.

Shine and Cosh (⋆)

The hyperbolic trig functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} \qquad \operatorname{cosech}(x) = \frac{1}{\cosh(x)} \qquad \coth(x) = \frac{1}{\tanh(x)}$$

(a) Show $\cosh^2(x) - \sinh^2(x) = 1$.

(b) Show $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$.

(c) Show $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$

(d) Show $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

(e) Show $\sinh(2x) = 2\sinh(x)\cosh(x)$

(f) Show that $1 - \tanh^2(x) = \operatorname{sech}^2(x)$

(g) Find a formula for $\tanh(x + y)$.

(h) Show the derivative of sinh is cosh.

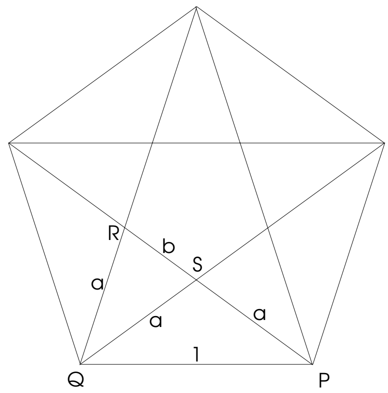
(i) Show the derivative of cosh is sinh.

(j) Show that $\sinh^{-1}(x) = \ln\left|x + \sqrt{x^2 + 1}\right|$.

(k) Show that the derivative of $\sinh^{-1}$ is $\dfrac{1}{\sqrt{x^2 + 1}}$.

(l) Show $\tanh(\ln|x|) = \dfrac{x^2 - 1}{x^2 + 1}$.

Diabolical (⋆)



A pentagram is inscribed inside a regular pentagon of side 1 as shown. By examining angles in this diagram it is possible to show that $\triangle PQR$ is similar to $\triangle QRS$. Show that the diagonals of the pentagon have length equal to the golden ratio

$$r = \frac{1+\sqrt{5}}{2}$$

Tinker Taylor (⋆ ⋆)

Taylor Series for a function

The Taylor series for a function $y = f(x)$ is the series

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

For many functions the Taylor series converges everywhere to $f(x)$. If this is true the function is said to be **analytic**. Many common functions are analytic.

(a) The functions $y = e^x$, $y = \cos(x)$ and $y = \sin(x)$ are all analytic. Express these functions in terms of their Taylor series.

(b) Using Taylor series allows us to extend the definition of analytic functions to complex numbers. If we define e^z, $\sin(z)$ and $\cos(z)$ in this way for complex z then show that

(i) $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

(ii) $\cos(ix) = \dfrac{e^x + e^{-x}}{2} = \cosh(x)$

(iii) $\sin(ix) = i\dfrac{e^x - e^{-x}}{2} = i\sinh(x)$

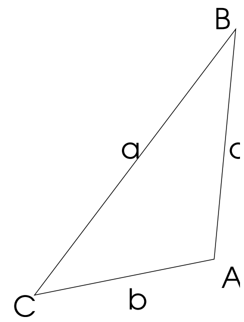
No Herons required (⋆ ⋆ ⋆)

Three point A, B and C are moving in the plane. The triangle between them has sides a, b and c as shown in the diagram

(a) Find an equation for the rate at which the angle A is changing in terms of a, b, c and their derivatives.

(b) Find a formula for the rate of change of the area of the triangle in terms of A b c and their derivatives.

(c) If at some instant the triangle is equilateral with sides of length 1, and the lengths of the sides a, b and c are increasing at 1ms^{-1}, 2ms^{-1} and 3ms^{-1} respectively, find the rate of change of the area.

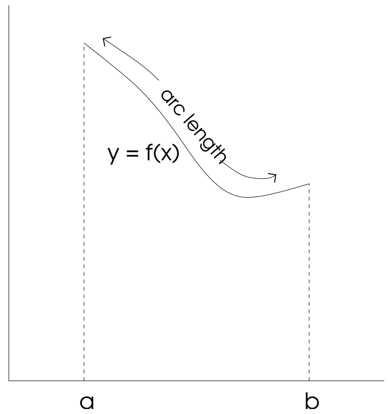


Is this Normal? (⋆ ⋆ ⋆)

Suppose that three points on the parabola $y = x^2$ have normal lines which intersect at a common point. Show that the sum of their x-coordinates is zero.

High Tension (⋆ ⋆)

The Arc Length Formula



The length of the curve with equation $y = f(x)$ between $x = a$ and $x = b$ is given by the arc length formula

$$\text{Length} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

A cable of length l is suspended between two towers of equal height a distance $2d$ apart, so that it sags a distance h in the centre.

The curve formed by a suspended rope or cable is called a **catenary**. Using a coordinate system with the lowest point of the catenary at the origin such a curve is described by the equation

$$y = \frac{1}{a}(\cosh(ax) - 1) = \frac{1}{2a}(e^{ax} + e^{-ax} - 2)$$

where a is a constant.

(a) Use the arc length formula to show that $l = \dfrac{e^{ad} - e^{-ad}}{a}$.

(b) Show that $(1 + ah)^2 = 1 + l^2$.

Up Up and Away (⋆ ⋆)

A Balloon is rising at the rate of 2ms^{-1}. When the balloon is 30 metres above the ground a cyclist moving at 5ms^{-1} passes directly underneath it. How fast is the distance $s(t)$ between the cyclist and the balloon increasing 3 seconds later.

Very Touching (⋆)

Find the coordinates of the point P on the curve $y = 2x^2 + \dfrac{8}{x}$ whose tangent passes through the origin. Also find the coordinates of the point Q where this tangent meets the curve again.

Imaginary Square (⋆ ⋆)

A quartic polynomial has complex roots on the corners of a square in the complex plane. One root is at $z = 1$ and the root diagonally opposite it is at $z = 4 + 4i$. Find the quartic polynomial assuming that the coefficient of z^4 is 1.

Cough Cough (⋆ ⋆)

When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the questions of how much it should contract to maximise the velocity and whether it really contracts that much when we cough. Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity v can be modelled by the equation

$$v = c(r_0 - r)r^2 \text{cm/sec} \ , \ \frac{r_0}{2} \leq r \leq r_0$$

where r_0 is the rest radius of the trachea in centimetres and and c is a positive constant whose value depends in part on the length of the trachea.

Show that v is greatest when $r = \frac{2}{3}r_0$, that is when the trachea is about 33% contracted. Remarkably X-rays confirm that the trachea really does contract about this much during a cough.

There and Back Again (⋆ ⋆)

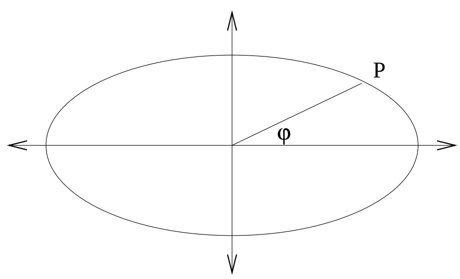
A particle moving along a straight line starts from position O at time $t = 0$ and moves in such a way that its displacement x from O at time t is $x = te^{-kt}$ where k is a positive constant. Show that its initial speed does not depend on k. Show also that the time taken from the start to reach the point of greatest displacement from O is half the time taken from the start to reach the greatest speed on the return journey towards O.

Elliptical Misconceptions (⋆ ⋆ ⋆)

In the parametric equation $(x, y) = (a\cos\theta, b\sin\theta)$ of an ellipse $(a \neq b)$, the parameter θ at point P is **not** the same as the angle ϕ at the origin, although it is often mistaken for this. The two angles θ and ϕ are however, closely related. In particular they are the same at the x and y intercepts.

(a) Write an equation relating θ and ϕ

(b) Find the points on the ellipse where the difference between the two angles is the greatest.



Seven into Three (⋆ ⋆ ⋆)

Given that α is a complex root of $x^7 - 1 = 0$, prove that the cubic equation whose roots are $\{\alpha + \alpha^6, \alpha^2 + \alpha^5, \alpha^3 + \alpha^4\}$ is $z^3 + z^2 - 2z - 1 = 0$.

Why Coordinates? (⋆ ⋆ ⋆)

The tangent to $y = f(x)$ at any point $x = a$ has y-intercept a. If the curve passes through $(1, 2)$ then find $f(x)$. *Hint: Substituting* $u = \frac{y}{x}$ *may help with the DE.*

Zeds (⋆)

Let $z_k = \cos\left(\frac{k\pi}{4}\right) + i\sin\left(\frac{k\pi}{4}\right)$. Find the exact value of

(a) $z_1 z_2 z_3 \ldots z_8$ **(b)** $z_1 + z_2 + z_3 + \ldots + z_8$

Tangent Time (⋆ ⋆)

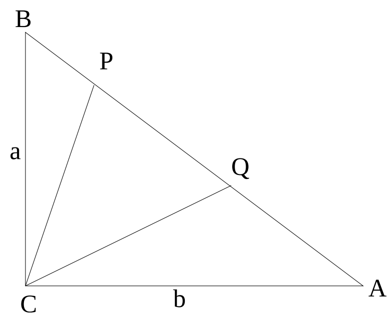
Find the equation of the tangent line to the curve $4y^3 = 27x^2$ at the point $(2t^3, 3t^2)$. Prove that perpendicular tangents intersect on the curve $y = x^2 + 1$.

Chord Short (⋆ ⋆)

Consider the lines through point $(2\sqrt{3}, 0)$ which cut both branches of the hyperbola $xy = 1$. Find the equation of the line which minimises the distance between the points where it cuts the hyperbola.

Area of Difficulty (⋆ ⋆ ⋆ ⋆)

Consider the right-angled triangle ACB as shown in the diagram. If $AP = AC = b$ and $BQ = BC = a$, find the area of $\triangle CPQ$ in terms of a and b. Also show that the angle PCQ is always $\frac{\pi}{4}$.



Integration by Wholes (⋆ ⋆)

(a) Show that $\displaystyle\int_0^{\pi} \sin^3(x)\,dx = \frac{4}{3}$.

(b) Use the substitution $u = \pi - x$ to show that $\displaystyle\int_0^{\pi} x\sin^3(x)\,dx = \frac{2\pi}{3}$.

Strangely Moving ($\star\star$)

A particle moves along a straight line in such a way that its velocity v and position x are related by the equation

$$v = \frac{a}{1+bx}$$

If the particle is at the origin at time $t = 0$, find the position of the particle at time t and show that accelleration of the particle is proportional to v^3.

Down To The Wire ($\star\star\star$)

A length of wire is cut into two parts, a square being formed of one part and a circle of the other. Find the ratio of these parts if the sum of the areas of the square and the circle is to be minimum.

Lets get Physical ($\star$)

An object moves along a straight line so that its velocity v and position x satisfy the differential equation

$$\frac{d}{dx}(v^2) = 20 - 10x$$

and satisfy the initial condition $v = 0$ when $x = 0$. At what other point is $v = 0$? What is the maximum velocity?

Vanishing Point ($\star\star$)

A curve has equation $y^3 = e^x y + e^{2x}$.

(a) Find an expression for $\dfrac{dy}{dx}$.

(b) If (a, b) is a point on the curve where $\dfrac{dy}{dx} = 0$, find the values of a and b.

Simply Complex ($\star\star$)

Let $w_k = \dfrac{k}{k+1} \operatorname{cis}\left(\dfrac{k\pi}{180}\right)$. Find $w_1 w_2 w_3 \ldots w_{179} w_{180}$

Off on a Tangent ($\star\star\star$)

At time $t = 0$ two particles are at point $A = (1, 0)$ on the circle with radius 1 centered at the origin. The first particle leaves the curve on a tangential path and moves upwards at the rate of 2 units per second. The second particle moves anticlockwise around the circle at the rate of 1 unit per second.

Find the location of the particles at time $t = \pi$ seconds, and determine the rate of change of the distance between them at this instant.

Conic Inter-section ($\star\star\star$)

The equation $x^2 - y^2 = 5x - y - 6$ describes a pair of lines. Find a separate equation for each line.

Verify This! ($\star\star$)

If $y = \dfrac{A\cos(mx) + B\sin(mx)}{x^n}$ show that

$$\frac{d^2y}{dx^2} + \frac{2n}{x}\frac{dy}{dx} + \left(\frac{n(n-1)}{x^2} + m^2\right) y = 0$$

Strangely Complex ($\star\star\star$)

Let u be a complex 16th root of 1.

(a) Find $1 + u + u^2 + u^3 + \ldots + u^{15}$.

(b) Find $1.u.u^2.u^3.\ldots.u^{15}$.

(c) Find $(1 + u)(1 + u^2)(1 + u^3)\ldots(1 + u^{15})$.

(d) Show that $(1 - u)(1 - u^9) = 1 - u^2$ and $(1 - u^2)(1 - u^{10}) = 1 - u^4$.

(e) Find $(1 - u)(1 - u^2)(1 - u^3)\ldots(1 - u^{15})$.

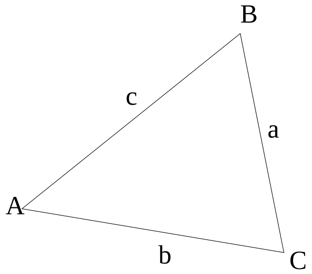
A Hyperbolic Norm ($\star\star\star$)

If the normal to the hyperbola $x = a\sec\theta,\ y = b\tan\theta$ at the point $\theta = \alpha$ meets the curve again at the point $\theta = \beta$, prove that

$$a^2\sin(\alpha) + b^2\sin(\beta) = (a^2 + b^2)\tan(\alpha)\cos(\beta)$$

Perimetric Form (⋆ ⋆ ⋆)

Consider a triangle with side lengths a, b, c and angles A, B, C.



(a) Show that $2bc(1 - \cos A) = (a + b - c)(a - b + c)$

(b) Obtain a similar expression for $2bc(1 + \cos A)$.

(c) Hence or otherwise prove Heron's formula for the area of a triangle $\text{Area}^2 = s(s-a)(s-b)(s-c)$ where $2s = a + b + c$

Cutting Cones (⋆ ⋆ ⋆ ⋆)

A solid right circular cone of base radius r and height h is drilled through the centre of its base along its axis until the cap is removed. What radius of drill-bit will ensure that the resulting solid has maximum total surface area?

The Power of Tan (⋆ ⋆ ⋆)

By using the identity $\tan^2(\theta) = \sec^2(\theta) - 1$, show that

$$\int \tan^{n+2}(\theta)\, d\theta = \frac{\tan^{n+1}(\theta)}{n+1} - \int \tan^n(\theta)\, d\theta$$

If $I_n = \int_0^{\frac{\pi}{4}} \tan^n(\theta)\, d\theta$, find I_0 and I_1. Hence give two formulae for I_n, one in the case that n is even and the other in the case that n is odd.

Tanking Up (⋆ ⋆ ⋆)

An tank in a brewery is shaped like a cylinder lying on its side. The radius of the cylinder is 1 metre and the cylinder is 4 metres long. The tank is being filled with liquid at the rate of 10 litres per second. At what rate is the depth of liquid increasing when the depth of liquid in the tank is 0.5 metres.

Angling for Answers (⋆ ⋆)

The curves with equations $y = 2x^2 - x$ and $y^2 = x$ intersect at the origin and at another point Q. Find the angle between the two curves at Q.

Around the Cone (⋆ ⋆)

A cone has height h and radius r. Let P be a point on the edge of the base of the cone. What is the length of the shortest possible path which travels right around the cone on the curved surface starting and ending at the point P.

Below the Line (⋆ ⋆ ⋆)

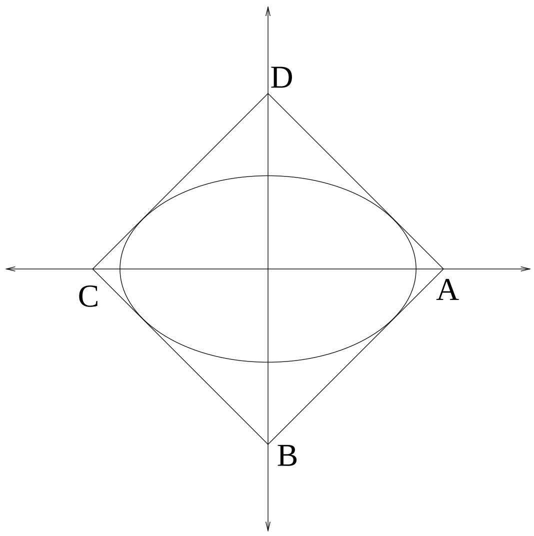
Show that the tangent to the curve $y = \sqrt{\sin x}$ at the point where $x = \frac{\pi}{3}$ lies above the curve for all values of x in the range $0 \leq x \leq \pi$.

1-2-3 Triangle (⋆ ⋆ ⋆)

The three sides of a triangle have lengths which are in an arithmetic sequence. By using the cosine rule (or otherwise) show that the angle opposite the edge which is neither greatest nor least is less than $\dfrac{\pi}{3}$.

Squaring The Ellipse (⋆ ⋆ ⋆)

The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ just fits inside the square $ABCD$. Find the area of the square.



Pythagorus meets Fibonacci (⋆ ⋆ ⋆)

A Pythagorean triple is a set of three whole numbers $\{a, b, c\} \subset \mathbb{N}$ that could form the sides of a right angled triangle. For example $\{3, 4, 5\}$ and $\{5, 12, 13\}$ are Pythagorean triples.

Consider any sequence of natural numbers satisfying the Fibonacci recursion law $t_{n+2} = t_n + t_{n+1}$ $(n \in \mathbb{N})$. Prove that $\{t_n t_{n+3},\ 2t_{n+1}t_{n+2},\ t_{n+1}^2 + t_{n+2}^2\}$ is always a Pythagorean triple.

Infinitely Integrable (⋆ ⋆ ⋆)

Simplify $\dfrac{1}{3}\left(\dfrac{4}{1+4u} - \dfrac{1}{1+u}\right)$. Hence find $\lim_{a\to\infty} \displaystyle\int_0^a \dfrac{x\,dx}{(1+x^2)(1+4x^2)}$.

A Right Tricky problem (⋆ ⋆ ⋆)

Find the minimum possible perimeter of a right angled triangle with an area of one square metre.

Evolution (⋆ ⋆)

Evolutes and the Centre of Curvature

The **centre of curvature** for a point $P = (x, y)$ on a curve is the centre $C = (\alpha, \beta)$ of the circle of curvature at P; the circle which most closely approximates the curve at P, and in particular has the same first and second derivatives. The coordinates of the centre of curvature are

$$(\alpha, \beta) = \left(x - \frac{dy}{dx}\left(\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right) \;,\; y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \right)$$

The **evolute** of a curve is the locus of all of its centres of curvature.

Show that the evolute of the parabola $y = x^2$ has equation $y = 3\left(\frac{x}{4}\right)^{\frac{2}{3}} + \frac{1}{2}$.

A Very Original Triangle (⋆ ⋆ ⋆ ⋆ ⋆)

The circle $x^2 + y^2 = 4a^2$ intersects the hyperbola with equation $xy = a^2$ at two points A and B in the first quadrant. Show that the triangle $\triangle OAB$ is equilateral where O denotes the origin.

A Supertantastic Identity (⋆ ⋆ ⋆)

(a) Show that, if $\theta_1 + \theta_2 + \cdots + \theta_n$ is a multiple of π then

$$(1 + i\tan(\theta_1))(1 + i\tan(\theta_2)) \cdots (1 + i\tan(\theta_n)) \text{ is real.}$$

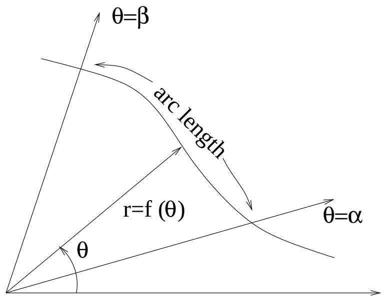
(b) Hence show that the interior angles A, B and C of a triangle obey the identity

$$\tan(A)\tan(B)\tan(C) - \tan(A) - \tan(B) - \tan(C) = 0$$

(c) Find a similar identity involving the interior angles of a quadrilateral. Or a pentagon!

Polar Circumnavigation (⋆ ⋆)

Arc length in Polar Coordinates



The arc length of the curve with polar equation $r = f(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ can be found using the formula

$$\text{Length} = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

(a) Apply this formula to the circle $r = a$, and verify that it gives the expected value for the circumference.

(b) Find the circumference of the cardioid with polar equation

$$r = a(1 - \cos(\theta))$$

Cubic Symmetry (⋆ ⋆ ⋆)

Suppose a cubic polynomial has a local minimum at $A = (x_1, y_1)$ and a local maximum at $B = (x_2, y_2)$. Show that the point of inflection is at the midpoint of AB.

Odd and Even (⋆ ⋆)

A function f is called *even* if $f(-x) = f(x)$ for all x in its domain, and *odd* if $f(-x) = -f(x)$. Prove that f' is odd when f is even and even when f is odd.

Elliptical Angles (⋆)

Show that the circle with equation $x^2 + y^2 = a^2 + b^2$ encloses the ellipse with equation $b^2x^2 + a^2y^2 = a^2b^2$.

Limited Growth Equation (⋆ ⋆ ⋆)

(a) Find A and B so that $\quad \dfrac{A}{x} + \dfrac{B}{b-x} = \dfrac{b}{x(b-x)}$

(b) Solve the differential equation $\dfrac{dx}{dt} = ax(b-x)$ where a and b are constant.

This differential equation can be used to model growth in situations where the maximum population is limited.

More Tan Tricks (⋆ ⋆ ⋆)

If $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n\theta \, d\theta$ show that $I_n = \dfrac{1}{n-1} - I_{n-2}$.

Hence or otherwise evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \left(\tan^6\theta + \tan^5\theta\right) \, d\theta$

The Can (⋆)

A cylindrical can of radius R has a marble of radius $r < R$ placed inside it. Water is then poured into the can until the marble is just submerged. If the marble is small not much water will be needed to submerge the marble. If the marble is big most of the can will be filled with marble and there will not be much room for water. Somewhere between these two extremes is a value of r that maximises the amount of water in the can. Find this value of r.

Holding it Steady (⋆)

Two concentric circles of radii $R > r$ trap a ring shaped area between them. The two radii are increasing in such a way that the area of the ring remains constant. If the larger radius is increasing at half the rate of the smaller radius when the smaller radius is 3, find the area of the ring.

Circular Arguments (⋆ ⋆)

Prove that, if $|z| = 1, z \in \mathbb{C}$ then $\arg\left(\dfrac{1+z}{1-z}\right) = \pm\dfrac{\pi}{2}$

Tricircle (⋆ ⋆)

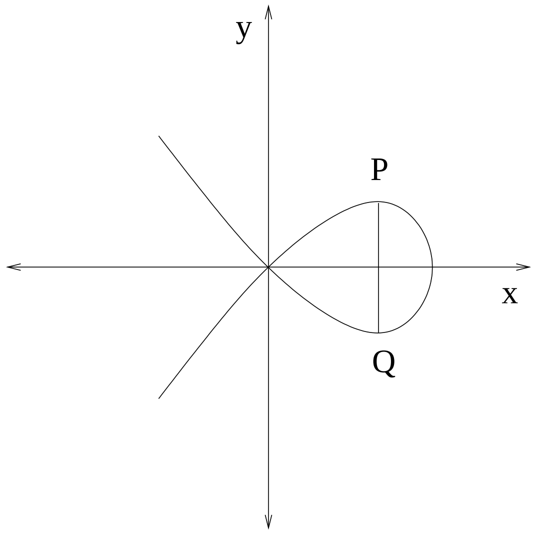
Three circles, one of unit radius and two others of radius r, touch each other tangentially in such a way that the lines joining their centres form a right-angled triangle. Find r, and show that the area enclosed by the circles, outside the circles, is $3 + 2\sqrt{2} - \dfrac{\pi}{2}(2+\sqrt{2})$ square units.

Hyperbolic Triangles (⋆ ⋆ ⋆)

Let P be any point in the first quadrant on the hyperbola $xy = 1$. The tangent to the hyperbola at P meets the x-axis at A and the y-axis at B. Show that the area of $\triangle AOB$ does not depend on the choice of P.

A Fishy Chord (⋆ ⋆)

PQ is a chord, parallel to the y axis, of the loop of the curve $y^2 = x^2(12-x)$. Calculate the maximum possible length of PQ.



Double Trouble (⋆ ⋆)

Find all solutions to the equation $\sin(2x)+\cos(2x)+\sin(x)+\cos(x)+1=0$ in the interval $0 \le x \le 2\pi$.

Tschirnhausen's cubic (⋆ ⋆ ⋆)

The curve with equation $y^2 = x^2(x+3)$ is called **Tschirnhausen's cubic**. If you graph this curve you will see that part of the curve forms a loop. Find the area enclosed by the loop.

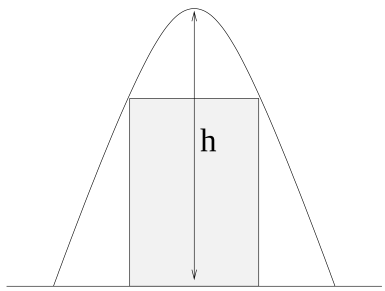
Nearest Distance (⋆ ⋆ ⋆)

Find the smallest possible distance between a point P on the parabola $y = x^2+5$ and a point Q on the line $2x+y=1$. Find the points P and Q which give this nearest distance.

A Normal Deduction (⋆ ⋆ ⋆)

C is a curve with parametric equations $x = \dfrac{2t}{1+t^2}$ and $y = \dfrac{1-t^2}{1+t^2}$. Find the normal to C at the point $t = a$. Hence or otherwise deduce the Cartesian equation of C.

Under the Arch (⋆ ⋆ ⋆)



An inverted parabola with height h has a rectangle of maximum possible area drawn beneath it. Show that the rectangle has height $\frac{2}{3}h$.

Sums to Products (⋆ ⋆ ⋆)

Find the general solution to the equation

$$\sin(x) + \sin(3x) = \sin(2x) + \sin(4x)$$

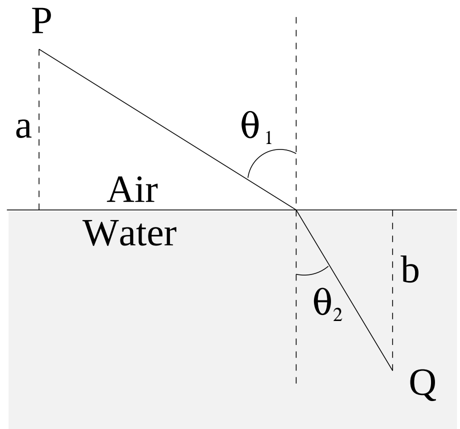
and hence find all solutions in the range $0 \leq x \leq 2\pi$.

Snell's Law (⋆ ⋆ ⋆)

Light travels from one point to another along the quickest path (the path that requires the least amount of time). Suppose that light has velocity v_1 in air and velocity v_2 in water, where $v_1 > v_2$. If light travels from point P in air to point Q in water along the quickest path, show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

This is Snell's law of refraction.



Direct Tricks (⋆ ⋆ ⋆ ⋆)

The parabola with equation $y^2 = 4ax$ has focus at $(a, 0)$ and has the vertical line $x = -a$ as directrix. A **focal chord** for a parabola is a line segment AB between two points on the parabola that passes through the focus. If AB is a focal chord of the parabola $y^2 = 4ax$, show that the tangent lines at A and B

(a) are perpendicular.

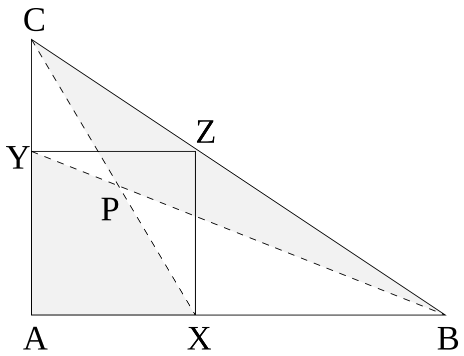
(b) intersect on the directrix.

Cubic Conundrum (⋆ ⋆ ⋆)

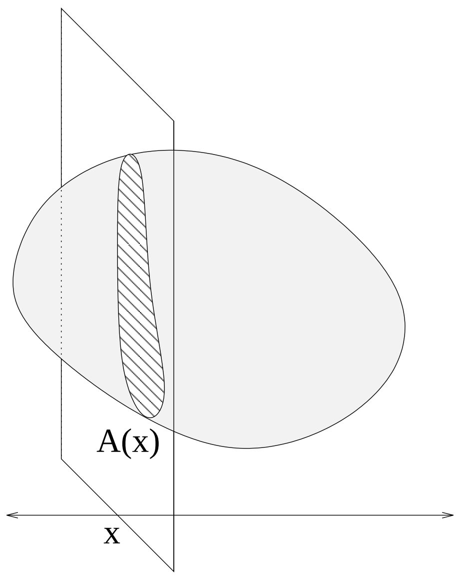
Two cubic graphs intersect, at right angles, at their point(s) of inflection. They are symmetrical so that $f(x) = g(-x)$ and $f(x) = -g(x)$. The total area enclosed by them is 1. What is the equation of either graph?

Quad Size (⋆ ⋆ ⋆)

A square is drawn inside the right angle triangle CAB with corners at X and Y as shown. Lines from X and Y to the corners of the triangle meet at P. Show that the area of quadrilateral $AXPY$ is the same as the area of triangle BPC.



The Cross Section (⋆ ⋆ ⋆ ⋆)



Volumes by Cross section

The volume of a 3-D solid is determined by the areas of cross sections perpendicular to an axis.

If the area of the cross section perpendicular to the x-axis at x is $A(x)$, then the volume of the solid is

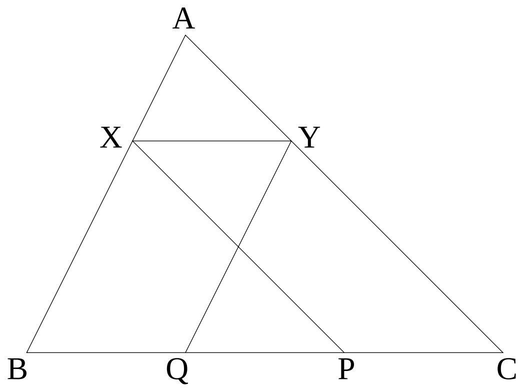
$$\int A(x)\, dx$$

(a) Two long cylindrical pipes of the same radius r intersect each other in such a way that their axes cross at right angles. Find the volume of the solid formed by their intersection.

(b) A cylinder of radius r and height h is cut by a plane tangent to the edge of the base and passing through the center of the top. In what ratio is the volume divided.

Triangularity (⋆ ⋆ ⋆)

In a triangle ABC the point X lies on AB between A and B. The straight line XY is drawn parallel to BC to meet AC at Y. The straight line XP drawn parallel to AC meets BC and P and the straight line YQ drawn parallel to AB meets BC at Q. Prove that the minimum value of $|XY|^2 + |PQ|^2$ is $\frac{1}{5}|BC|^2$.



Free Wheeling (⋆ ⋆ ⋆)

The deceleration of a free wheeling cyclist on a flat piece of road with no wind, is proportional to velocity. A cyclist travelling at initial speed u starts free wheeling and decelerates over time t to a speed of v. If d is the distance travelled show that

$$\ln\left(\frac{u}{v}\right) = \frac{t(u-v)}{d}$$

A Most Particular Quarticular (⋆ ⋆ ⋆)

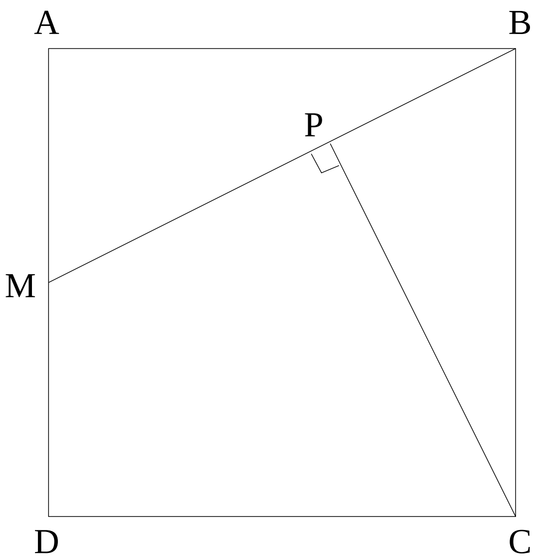
The quartic polynomial $x^4 - 8x^3 + 19x^2 + kx + 2$ has four distinct real roots denoted a, b, c and d; in order from smallest to largest. If $a + d = b + c$ then

(a) show that $a + d = b + c = 4$

(b) show that $abcd = 2$ and $ad + bc = 3$

(c) find ad and bc.

(d) find a, b, c, d and k.

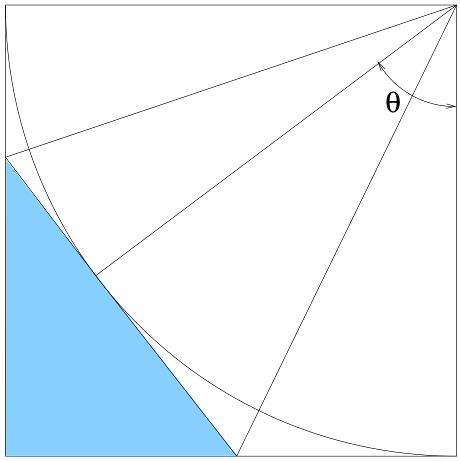


Simple Geometry (⋆ ⋆ ⋆)

$ABCD$ is a square. M is the mid-point of AD. The line from C perpendicular to MB meets MB at P. Prove that DP and DC have the same length.

This question is indeed simple if you spot the trick. It can be solved with a flash of insight and a better diagram.

Calculating Corners (⋆ ⋆ ⋆)



A quarter circle is drawn inside a square of side length 1 as shown. A radius is drawn having angle θ to one side of the square, and a tangent line is drawn at the point where this radius meets the circumference of the quarter circle. Show that the area of the resulting triangle is

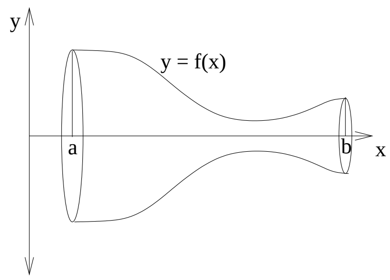
$$A = \frac{t(1-t)}{1+t} \quad \text{where} \quad t = \tan\left(\frac{\theta}{2}\right)$$

Quad-Sin-Cos (⋆)

Find all solutions to the equation $\sin^2(x) + \cos(x) + 1 = 0$ in the interval $0 \le x \le 2\pi$.

Surface (⋆ ⋆)

Surface of Revolution



The area of the curved surface obtained by rotating the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ around the x-axis is given by the following formula.

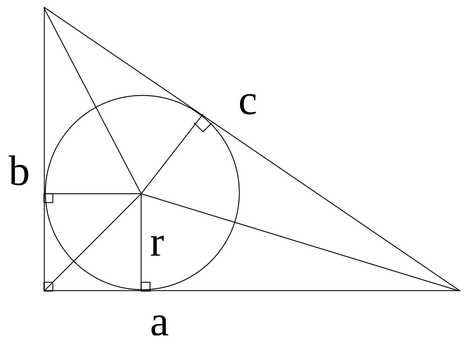
$$\text{Area} = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx$$

(a) Apply this formula to the constant function $y = r$ between $x = 0$ and $x = h$. Verify that it gives the expected value for the curved surface area of a cylinder.

(b) Apply this formula to the function $y = \frac{r}{h}x$ between $x = 0$ and $x = h$ and verify that it gives the expected value for the curved surface area of a cone.

(c) Use the formula to compute the curved surface area of a sphere.

Right Round Circle (⋆ ⋆ ⋆)



The radius of the inscribed circle of this right-angled triangle is r.

(a) Prove that $r = \dfrac{ab}{a+b+c}$

(b) Show that $r = \dfrac{a+b-c}{2}$

Top of the Pops (⋆ ⋆ ⋆)

(*requires some knowledge of the physics of projectile motion*)

Projectiles are fired with initial speed v and variable launch angle $0 < \alpha < \pi$. Choose a coordinate system with the firing position at the origin. For each value of α the trajectory will follow a parabolic arc with apex at (x, y) where both x and y depend on α. Show that

$$x^2 + 4\left(y - \frac{v^2}{4g}\right)^2 = \frac{v^4}{4g^2}$$

and hence the points of maximum height of the trajectories lie on an ellipse.

A Problem of Icosahedral Dimensions (⋆ ⋆ ⋆)

A regular icosahedron can be placed with its 12 vertices at coordinates

$$\{(0, \pm 1, \pm r), (\pm 1, \pm r, 0), (\pm r, 0, \pm 1)\}$$

providing the constant r is chosen carefully.

(a) Find the value of r needed to make this work.

(b) What is the volume of this icosahedron.

(c) Use this to find a formula for the volume of an icosahedron of side length a.

(d) Can you find the formula for the volume of a dodecahedon?

A Tantalizing Problem (⋆ ⋆ ⋆)

If A, B and C are the angles in a triangle, show that

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = k$$

where k is the same for all triangles.

Sometimes Not ($\star$)

(a) Show that $\dfrac{\sin(\theta)+\sin(2\theta)}{1+\cos(\theta)+\cos(2\theta)} = \tan(\theta)$ where $0 \le \theta < \frac{\pi}{2}$.

(b) Find the values of θ where this identity is NOT true where $\frac{\pi}{2} < \theta \le \pi$.

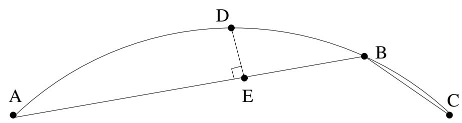
Linearity ($\star\star$)

Let f be a function on real numbers with the property that $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. Assume that the derivative of f at zero exists and let $f'(0) = k$.

Show that the derivative $f'(x)$ exists at all $x \in \mathbb{R}$.

Hence or otherwise prove that $f(x) = kx$.

Broken Chord ($\star\star\star$)



In this diagram let D be the midpoint of the arc $ADBC$. If DE is perpendicular to AB show that

$$|AE| = |EB| + |BC|$$

This result is known as Archimedes theorem of the broken chord.

Closely Rooted ($\star\star$)

Consider the equation $x^2 + (3a-2)x + (a^2+4a-9) = 0$.

(a) Show that for all values of a this quadratic in x has two distinct real roots.

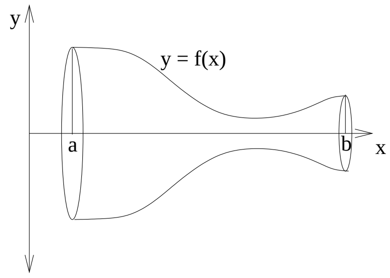
(b) Find the value of a where the two roots are the closest together. What is the value of this smallest distance between roots?

Trajectory ($\star\star\star\star\star$)

A model aeroplane flies near two observers standing in a flat field in such a way that at any instant the angle of elevation seen by observer A is exactly twice the angle of elevation seen by observer B. Using a coordinate system with A at $(-1, 0)$ and B at $(1, 0)$ show that this condition confines the plane to fly above a circular region on the field containing observer A. Find the center and radius of this circular region and give a formula in terms of x and y for the height h of the plane at any point (x, y) inside it.

Volume (⋆ ⋆)

Volume of Revolution



The volume obtained by rotating the area under the curve $y = f(x)$ between $x = a$ and $x = b$ around the x-axis is given by the following formula.

$$\text{Volume} = \pi \int_a^b y^2\,dx = \pi \int_a^b (f(x))^2\,dx$$

(a) Apply this formula to the constant function $y = r$ between $x = 0$ and $x = h$. Verify that it gives the expected value for the volume of a cylinder.

(b) Apply this formula to the function $y = \frac{r}{h}x$ between $x = 0$ and $x = h$ and verify that it gives the expected value for the volume of a cone.

(c) Use the formula to compute the volume of a sphere.

Just In the Area (⋆)

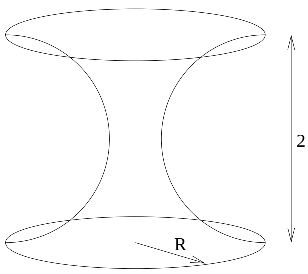
This question requires volumes of revolution. See the infobox on page 46.

Sketch the curve $y^2 = \dfrac{x}{2-x}$ and find the area bounded by the curve, the x-axis, and the line $x = 1$. Show that the volume obtained by rotating this area around the x-axis is $\pi(2\ln(2) - 1)$.

Spool (⋆ ⋆ ⋆ ⋆)

This question requires volumes of revolution. See the infobox on page 46.

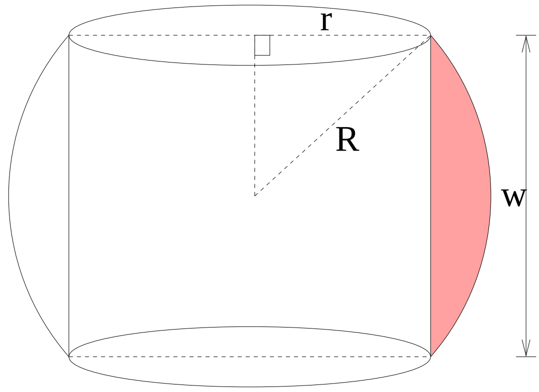
A spool is to be made by rotating a semicircle of radius 1 about a line a distance R away and parallel to its axis as depicted. Note that the spool will fit snugly against a sphere of radius 1. For what value of R will the spool have the same volume as the sphere?



The Rings of the Lord (⋆ ⋆ ⋆ ⋆)

This question requires volumes of revolution. See the infobox on page 46.

Lord Elrond of Rivendell has a collection of solid gold rings for different-sized fingers. The cross section of each ring is a segment of a circle or radius R as shown in the diagram. All rings in the collection have the same width w.



Lord Elrond Tells Gandalf that, although the rings have different diameters, they all contain the same amount of gold. Is this true? Justify your answer.

Turn Table (⋆ ⋆ ⋆)

This question requires volumes of revolution. See the infobox on page 46.

A cylindrical tank of radius 1 metre with its axis vertical is mounted on a turntable so that it can be spun around its axis. The tank is initially filled with water to a depth d. When the turntable is spun at a constant speed the surface of the water becomes parabolic in shape. If the speed is such that the paraboloid formed by the surface just touches the bottom of the tank, how far up the sides of the tank does the water go.

A Boring Problem (⋆ ⋆ ⋆ ⋆ ⋆)

This question requires volumes of revolution. See the infobox on page 46.

A circular hole is to be bored through the centre of a sphere. If the radius of the sphere is R, what radius r should be used for the drill bit so that exactly half of the volume of the sphere is cut away?

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How should I approach solving problems?

Understand the Problem : To begin make sure you fully understand the problem including what you have to do in order to have solved it.

Draw Pictures : A picture is worth a thousand words. Engage your visual processor. It is one of the most powerful parts of your brain.

Formulate : Formulate and choose suitable notation.

Plan : Think about how a solution to the problem might proceed.

Play : If you can't see what to do just play with the problem and see what happens. You might notice something useful.

Persist Some of these problems are long and you will need to stick at it and keep going through some difficult algebra.

Progress Regularly check if you are making progress. If not you might need to try a different approach.

Keep an Open Mind : Inspiration could strike at any time.

Avoid Fear : Fear shuts down your ability to think. Usually problems are not as difficult as they look once you get into them.

Have Fun : Try to approach the examination as an interesting enjoyable puzzle. Your brain is going to function better in this frame of mind.
